Engineering Analysis

using Scilab and C

Version 0.2 (August 2006)

by Hugh Jack

Engineer On a Disk

Overview: This note set is part of a larger collection of materials available at http://claymore.engineer.gvsu.edu. You are welcome to use the material under the license provided at http://claymore.engineer.gvsu.edu/eod/global/copyrght.html. As always any feedback you can provide will be welcomed.

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EGR 600 – Analysis
Summer 2006

Description: Undergraduate mathematics topics are reviewed and then extended to solve advanced engineering problems. The course will focus on solving problems such as project management, economic justification, modeling random processes, risk analysis, and system modeling. The course will make extensive use of computers in an active learning environment.

Pre-requisite: Admission to MSE or permission of instructor

Texts: ***Jack, H.,*** Engineering Analysis, 2006.

Software: Scilab (www.scilab.org) or Matlab
C/C++ compiler

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Office hours: 5:00-6:00pm Mondays
Class time: 6:00-8:50pm Mondays

Instruction Methods: Lecture, discussion, assignments.

Homework: Weekly homework problems will be assigned to reinforce concepts. In some cases students will be assigned alternate homework problems specific to their discipline.

Quizzes: Quizzes will be given frequently to assess pre-lecture reading and topic mastery.

Exams: A midterm and comprehensive final will be used to assess student learning.

Semester Project: Students will work individually on a semester project that explores an advanced area of engineering analysis. The project report will emphasize the need for the clear communication of mathematical concepts including a written report and an oral presentation.

Computer Use: Students will be expected to use computers to analyze engineering problems. Scilab will be the preferred computational platform, however Matlab may also be used. Students may be required to write programs in C or C++.

Grading Policies:

Homework and Quizzes	50%
Midterm Exam(s)	15%
Final Exam	35%

Note: Students must earn passing grades in all components of the course in order to receive a passing grade for the course.

Grading Scale:

A 80 – 100
B 70 - 80
C 60 - 70
D 55 - 60
F 0 – 55

Disability Services: If there is any student in this class who has special needs because of a learning, physical, or other disability, please contact me of the Office of Academic Support (OAS) at 331.2490.

Tentative Topics:

1. Mathematics Review and introduction to Scilab
    - Scilab Tutorial
    - Chapter - Numerical Values and Units
    - Chapter - Algebra
    - Chapter - Trigonometry
2. Programming and Data Presentation
    - Chapter - Graphing
    - Chapter - (PROGRAMMING??)
3. Probability and statistics applications
    - Chapter - Permutations and Combinations
    - Chapter - Probability
4. Calculus applications
    - Chapter - Calculus

5. Differential equations
    - Chapter - Differential Equations
6. Economics
    - Chapter - Financial
7. General emphasis specific review
    - Chapter - (by discipline)
    - Chapter - Boolean
    - Chapter - Transforms
8. Optimization
    - Chapter - Optimization
9. System reliability
    - Chapter - Reliability
10. Project scheduling
    - Chapter - Projects
11. Directed graphs
    - Chapter - Graphs
    - Chapter - Trees

27. PREFACE

- the book is designed for use for entry level graduate students, or working professionals looking for review/topical introduction.
- computer based solutions are used liberally throughout the book with a bias towards practical problems.
- basic book design - short chapters, easily used as one or more modules per class
- review modules are marked with the title review. This allows review material to be reviewed without having it dominate the book
- programs are written in C and Scilab. Scilab has been used because it is available freely, but is very similar to matlab
- example programs are boxed
- sample problem solutions are boxed
- asides and notes are given as appropriate
- the text is written to be very direct and student friendly in tone.
- written from an engineers perspective - using mathematics as a tool
- chapters begin and end with summaries
- in some cases topics are presented out of order. Although this would be a major problem if it was a first introduction, as a review it is done in the interests of brevity.

27.1 Todo

- create a map of techniques for solving specific problems.
- add a symbolic algebra package

Topics to Add/expand ??--

analysis of constrained path systems
Discrete math

Algebra

power series and Taylor series

Data

graphing data
error estimation
linear regression
curve fitting with splines
data representation graphs, tables and accuracy

- statistical analysis of a system

Random Processes
- random number generation
- prediction of process outcome
- Quality control

System Modeling
- (ME) state space analysis
- (ME) Eigenvalues and vectors

Advanced Calculus
- (ME) Partial differential equations
- (ECE) Laplace transforms
- Convolution
- Fast Fourier transforms
- differential equations, state space and numerical integration

Optimization
- (PDM) Searching convex/concave functions
- (MO) Searching convex/concave functions
- (MO) Gradient descent
- (MO) Cost and penalty functions

Programming
- programs - basic structure and execution
- program entry points
- syntax
- Variable names
- variable/data types
- input
- outputs
- logical expression
- conditional execution with ifs
- for loops and conditions
- functions/subroutines
    - input/output argument lists
    - defining a function
    - calling a function and dealing with return values
    - the difference between calling and defining a function

Discipline Specific
- ECE
    - Frequency domain - Fourier, Bessel
    - EM Theory review
    - Feedback Control Theory
    - 3 phase electrical theory

27.2 Problems Not Sorted

12a. Find y(t).

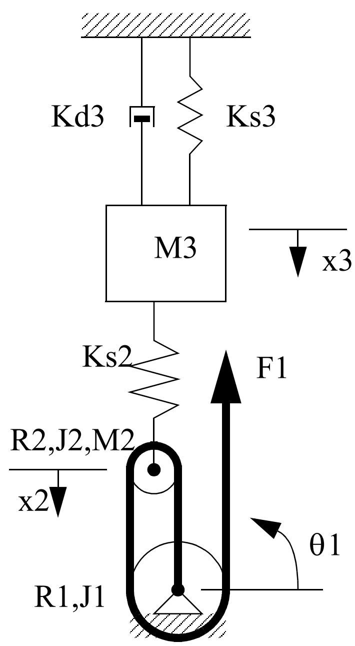
$$\frac{y(s)}{x(s)} = \frac{s^2 + 4s}{s^2 + 6s + 9} \qquad x(t) = 5$$

12b. Find x(t).

$$\ddot{x} + 4\dot{x} + 2x = 3 \qquad x(0) = 1 \qquad \dot{x}(0) = 0$$

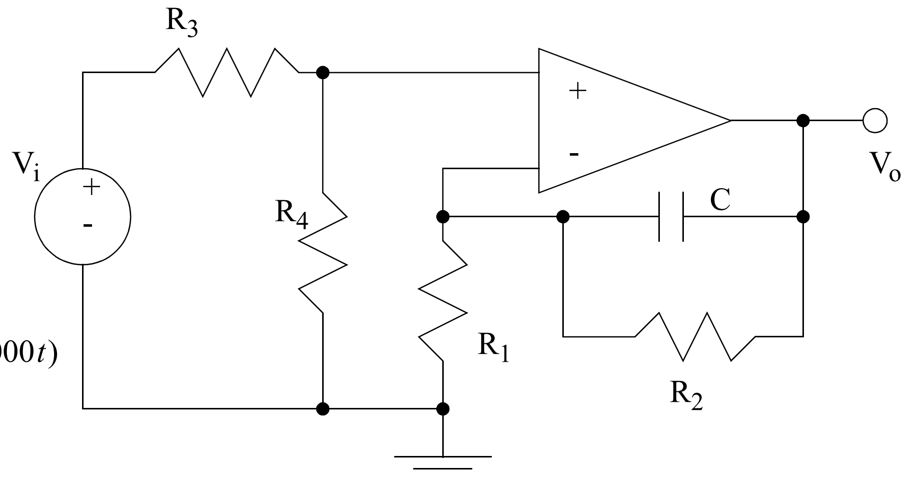
12c. An existing manual production line costs $100,000 to operate per year. A new piece of automated equipment is being considered to replace the manual production line. The new equipment costs $150,000 and requires $30,000 to operate. The decision to purchase the new machine will be based on a 3 year period with a 25% interest rate. Compare the present value of the two options.

14. For the system pictured below put the equations in state variable form and simulate the system using numerical integration. Assume values to test your program.



A pulley system has the bottom pulley anchored. A mass is hung in the middle of the arrangement with springs and dampers on either side. Assume that the cable is always tight.

15. Find the steady state outputs for the system using the given input functions.



$R_1 = R_2 = R_3 = R_4 = 1$Kohm
C=1uF

a) $V_i = 5\sin(100t)$

b) $V_i = 5\sin(1000000t)$

c) $V_i = 5$

28. NUMERICAL VALUES AND UNITS

Topics:
- Numbers, Constants, Units

Objectives:
- To review the use of basic numbers, including significant figures
- To review the use of units to keep track of numerical magnitudes

28.1 Introduction

28.2 Numerical Values

28.2.1 Constants and Other Stuff

• Some basic definitions,

numeric - a literal numerical value

variable - a symbol used to represent a quantity that will change, often represented with a lower case symbol

constant - a value that will not change, often represented with an upper case symbol

subscripts - letters or numbers below a variable to create new (related) variables.

• greek letters are often used for variables and constants

lower case	upper case	name
α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ε	E	epsilon
ζ	Z	zeta
η	H	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Y	upsilon
ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

• The constants listed are some of the main ones, other values can be derived through calculation.

$$e = 2.7182818\ldots = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = \text{natural logarithm base}$$

$$\pi = 3.1415927\ldots = \text{pi}$$

$$\gamma = 0.57721566 = \text{Eulers constant}$$

$$1\,radian = 57.29578°$$

- In Scilab

```
// basic variables and constants
a = 5; // define a variable with a value of 5
b = %pi; // the value for pi is assigned to b
c = %e; // the natural number
d = %inf; // infinity
e = %nan; // not a number
m = %t; // a logical true
n = %f; // a logical false
p = %i; // the imaginary number
q = eps; // a very small positive number, or 0+

abs(x); // returns the magnitude of x
int(x); // converts a real to an integer value


// getting information
who // print all variables
help sin // open a help window for the sin function
help + // get help on basic operators
apropos imaginary // look for functions on imaginary numbers
```

28.2.2 Factorial

- A compact representation of a series of increasing multiples.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

$$0! = 1$$

28.2.3 Significant Figures

- Sig figures rules,

- leading zeroes do not count as significant figures.
- trailing zeros will count as significant figures.
- when doing multiplying the results should will (generally) have the same number of significant figures as the least significant number.
- when adding, the least accurate number determines the accuracy of the result.

123.456	6 significant figures
123.45600	8 significant figures
0.000345	3 significant figures
$123 \cdot 456789 = 56185047 = 56.2 \times 10^3$	3 significant figures
$0.12(3456) = 414.72 = 0.41 \times 10^3$	2 significant figures
$34 + 56.789 = 90.789 = 91$	2 significant figures

• In computation the standard is to keep all of the digits, but the final answer should be rounded to the correct number of significant figures

• Based upon the accuracy of most measuring instruments, and the ability to specify components, most engineering calculations will have 3-6 significant figures. Do not use all of the digits produced by computer/calculator unless all of the digits can be justified.

28.2.4 Scientific and Engineering Notations

• In scientific notation one digit is ahead of the decimal, and all other values follow the decimal. The exponent is adjusted accordingly.

$$a = 1234.56789 = 1.23456789 \times 10^3 = 1.23456789e3$$

- Scilab

```
a = 1234.5678912345678;
a // by default 8 digits will be printed
format('v', 20) // set the number of displayed characters to 20
a // prints 1234.567891234567892
format('v', 5)
a // prints 1234.
format('e', 8) // set the display to exponent notation
a // prints 1.2D+03
a = 1.23456789123456789e3; // the same value of a entered in exponent format
```

- Engineering notation is similar to scientific notation, but the exponent is always a multiple of 3 so that it corresponds to magnitude multipliers (i.e., micro, milli, kilo, mega).

$$a = 12345.6789 = 12.3456789e3$$

$$b = 0.000123456789 = 123.456789e-6$$

- The current version of Scilab does not seem to support engineering notation.

28.3 Complex Numbers

- 'j' will be the preferred notation for the complex number, this is to help minimize confusion with the 'i' used for current in electrical engineering.

- The basic algebraic properties of these numbers are,

The Complex (imaginary) Number:

$$j = \sqrt{-1} \qquad\qquad j^2 = -1$$

• Scilab,

```
j = sqrt(-1); // define j as the imaginary number
A = 5 + 3*j; // define a complex number
B = 7 + 9j; // define another
A / B // a complex operation
```

28.4 Units and Conversions

• Units are essential when describing real things.

• Good engineering practice demands that each number should always be accompanied with a unit.

28.4.1 How to Use Units

• This section does not give an exhaustive list of conversion factors, but instead a minimal (but fairly complete) set is given. From the values below most conversion values can be derived.

• A simple example of unit conversion is given below,

**a simple unit conversion example:

Given,

$$d_x = 10m \qquad d_y = 5ft$$

Find the distance 'd',

keep the units in the equation

$$d = \sqrt{d_x^2 + d_y^2}$$

$$\therefore d = \sqrt{(10m)^2 + (5ft)^2}$$

multiply by 1

$$\therefore d = \sqrt{100m^2 + 25ft^2}$$

$$\therefore d = \sqrt{100m^2 + 25ft^2\left(\frac{0.3048m}{1ft}\right)^2}$$

$$\therefore d = \sqrt{100m^2 + 25ft^2(0.092903)\frac{m^2}{ft^2}}$$

cancel out units

$$\therefore d = \sqrt{100m^2 + 25(0.092903)m^2}$$

$$\therefore d = \sqrt{102.32m^2} = 10.12m$$

From the tables

$$1ft = 0.3048m$$

$$\therefore 1 = \frac{0.3048m}{1ft}$$

28.4.2 SI Units

1. Beware upper/lower case letter in many cases they can change meanings.
    e.g. m = milli or mega?
2. Try to move prefixes out of the denominator of the units.
    e.g., N/cm or KN/m
3. Use a slash or exponents.
    e.g., (kg•m/s^2) or (kg•m•s^{-2}) or (kg m s**-2) or (kg m s ^-2)
4. Use a dot in compound units when possible.
    e.g., N•m
5. Use spaces to divide digits when there are more than 5 figures, commas are avoided because their use is equivalent to decimal points in many places (e.g., Europe).

• Base and derived units

Base units

m = length

kg = mass

s = time

A = current

K = temerature

mol = chemical quantity

cd = candela

Derived unit examples

$$N = \frac{kg \cdot m}{s^2} \qquad F = \frac{C}{V}$$

$$J = N \cdot m \qquad \Omega = \frac{V}{A}$$

$$Pa = \frac{N}{m^2} \qquad Wb = V \cdot s$$

$$W = \frac{J}{s} \qquad T = \frac{Wb}{m^2}$$

$$V = \frac{W}{A} \qquad H = \frac{Wb}{A}$$

• In some cases units are non-standard. There are two major variations US units are marked with 'US' and Imperial units (aka English and inch based) are marked with 'IMP'.

28.4.3 A Table

Distance

1 ft. (feet) = 12 in. (inches) = 0.3048 m (meter)
1 mile = 1760 yards = 5280 ft = 1.609km
1 in.(inch) = 2.540 cm
1 yd (yard) = 3 ft.
1 nautical mile = 6080 ft. = 1852 m = 1.150782 mi
1 micron = 10^{-6} m
1 angstrom = 10^{-10} m

1 mil = 10^{-3} in
1 acre = 43,560 ft. = 0.4047 hectares
1 furlong = 660 ft
1 lightyear = 9.460528e15 m
1 parsec = 3.085678e16 m

Area

1 acre = 43,559.66 ft^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2
1 Hectare (ha) = 10,000 m^2

Velocity

1 mph = 0.8689762 knot

Angle

1 rev = 2PI radians = 360 degrees = 400 gradians
1 degree = 60 minutes
1 minute = 60 seconds

Volume

1 US gallon = 231 in^3
1 CC = 1 cm^3
1 IMP gallon = 277.274 in^3
1 barrel = 31 IMP gal. = 31.5 US gal.
1 US gal. = 3.785 l = 4 quarts = 8 pints = 16 cups
1 liter (l) = 0.001 m^3 = 2.1 pints (pt) = 1.06 quarts (qt) = 0.26 gallons (gal)
1 qt (quart) = 0.9464 l
1 cup (c) = 0.2365882 l = 8 USoz
1 US oz = 8 US drams = 456.0129 drops = 480 US minim = 1.0408 IMP oz
= 2 tablespoons = 6 teaspoons
1 IMP gal. = 1.201 U.S. gal.
1 US pint = 16 US oz
1 IMP pint = 20 IMP oz
1tablespoon = 0.5 oz.
1 bushel = 32 quarts
1 peck = 8 quarts

Force/Mass

1 N (newton) = 1 kg•m/s^2 = 100,000 dyne
1 dyne = $2.248*10^{-6}$ lb. (pound)
1 kg = 9.81 N (on earth surface) = 2.2046 lb

1lbf = 16 oz. (ounce) = 4.448N
1 oz. = 28.35 g (gram) = 0.2780N
1 lb = 0.03108 slug
1 kip = 1000 lb.
1 slug = 14.59 kg
1 imperial ton = 2000 lb = 907.2 kg
1 metric tonne = 1000 kg
1 troy oz = 480 grain (gr)
1 g = 5 carat
1 pennyweight = 24 grain
1 stone = 14 lb
1 long ton = 2240 lb
1 short ton = 2000 lb

Pressure

1 Pascal (Pa) = 1 N/m^2 = 6.895 kPa
1 atm (metric atmos.) =760 mmHg at 0°C=14.223 lb/in^2=1.0132*10^5 N/m^2
1 psi = 2.0355 in. Hg at 32F = 2.0416 in. Hg at 62F
1 microbar = 0.1 N/m^2

Scale/Magnitude

atto (a) = 10^{-18}
femto (f) = 10^{-15}
pico (p) = 10^{-12}
nano (n) = 10^{-9}
micro (μ) = 10^{-6}
milli (m) = 10^{-3}
centi (c) 10^{-2}
deci (d) = 10^{-1}
deka (da) = 10
hecto (h) = 10^2
kilo (K) = 10^3
mega (M) = 10^6
giga (G) = 10^9
tera (T) = 10^{12}
peta (P) = 10^{15}
exa (E) = 10^{18}

Power

1 h.p. (horsepower) = 745.7 W (watts) = 2.545 BTU/hr. = 550 ft.lb./sec.
1 ft•lb/s = 1.356 W
1 J (joule) = 1 N•m = 10^7 ergs = 0.2389 cal.

1 W = 1 J/s
1 ev = $1.60219*10^{-19}$ J
1 erg = 10^{-7} J

Temperature

°F = [(°C*9)/5]+32, °C = Celsius (Centigrade), F = Fahrenheit
K = Kelvin
Rankine (R) = F - 459.666
0.252 calories = 1 BTU (British Thermal Unit)
-273.2 °C = -459.7 °F = 0 K = 0 R = absolute zero
0 °C = 32 °F = 273.3 K = 491.7 R = Water Freezes
100°C = 212°F = 373.3 K = 671.7 R = Water Boils (1 atm. pressure)
1 therm = 100,000 BTU

Mathematical

π radians = 3.1416 radians = 180 degrees = 0.5 cycles
1 Hz = 1 cycle/sec.
1 rpm (revolutions per minute) = 60 RPS (Revolutions per second) = 60Hz
1 fps (foot per second) = 1 ft/sec
1 mph (miles per hour) = 1 mi./hr.
1 cfm (cubic foot per minute) = 1 ft^3/min.
e = 2.718

Time

1 Hz (hertz) = 1 s^{-1}
1 year = 365 days = 52 weeks = 12 months
1 leap year = 366 days
1 day = 24 hours
1 fortnight = 14 days
1 hour = 60 min.
1 min = 60 seconds
1 millenium = 1000 years
1 century = 100 years
1 decade = 10 years

Physical Constants

R = 1.987 cal/mole K = ideal gas law constant
K = Boltzmann's constant = $1.3x10^{-16}$ erg/K = $1.3x10^{-23}$ J/K
h = Planck's constant = $6.62x10^{-27}$ erg-sec = $6.62x10^{-34}$ J.sec
Avagadro's number = $6.02x10^{23}$ atoms/atomic weight
density of water = 1 g/cm^3
electron charge = $1.60x10^{-19}$ coul.
electron rest mass = 9.11*10**-31 Kg

proton rest mass = 1.67*10**-27 Kg
speed of light (c) = 3.00×10^{10} cm/sec
speed of sound in dry air 25 C = 331 m/s
gravitational constant = 6.67*10**-11 Nm**2/Kg**2
permittivity of free space = 8.85*10**-12 farad/m
permeability of free space = 1.26*10**-6 henry/m
mean radius of earth = 6370 Km
mass of earth = 5.98*10**24 Kg

Electromagnetic

magnetic flux = weber (We) = 10**8 maxwell
inductance = henry
magnetic flux density = tesla (T) = 10**4 gauss
magnetic intensity = ampere/m = 0.004*PI oersted
electric flux density = coulomb/m**2
capacitance = farad
permeability = henry/m
electric field strength = V/m
luminous flux = lumen
luminance = candela/m**2
1 flame = 4 foot candles = 43.05564 lux = 43.05564 meter-candles
illumination = lux
resistance = ohm

28.5 Problems

1. Show the units for Joules in base units. (ans. kg m^2 / s^2)

2. How many kips are in 2.00 metric tonnes. (ans. 4.41 kip)

3. Do the following calculation using significant figures.

$$0.010 \times 1.2345 + 0.0234567 \qquad \text{(ans. 35e-3)}$$

2. Simplify the following expressions.

a) $(6 + 8j)^2$ ans. $-28 + 96j$

b) $\dfrac{8j + 6}{(4j + 3)^2}$ ans. $\dfrac{6}{25} - \dfrac{8}{25}j$

c) $2 + 5!$ ans. 122

29. ALGEBRA

Topics:

-

Objectives:

-

29.1 The Fundamentals

29.1.1 Basic Operations

commutative	$a + b = b + a$	
	$a + b = c + d$	$\therefore a = c + d - b$
distributive/collective	$a(b + c) = ab + ac$	
associative	$a(bc) = (ab)c$	$a + (b + c) = (a + b) + c$
	$ab = cd$	$\therefore a = \frac{cd}{b}$
silly mistakes	$\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}$	

• Scilab example,

```
b = 5;
c = 6;
d = 7;
a = c * d / b
a = c + d - b
```

29.1.2 Exponents

• The basic properties of exponents are so important they demand some sort of mention

$(x^n)(x^m) = x^{n+m}$	$x^0 = 1$, if x is not 0	$x^{\frac{1}{n}} = \sqrt[n]{x}$ (nth root)
$\frac{(x^n)}{(x^m)} = x^{n-m} = \frac{1}{x^{m-n}}$	$x^{-p} = \frac{1}{x^p}$	$x^{\frac{m}{n}} = \sqrt[n]{x^m}$
$(x^n)^m = x^{n \cdot m}$	$(xy)^n = (x^n)(y^n)$	$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$
	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	

• Scilab example,

```
x = 5;
n = 2;
m = 4;
y = 6;
x^n * x^m
x^(n + m)
x**(n+m)
```

29.1.3 Basic Polynomials

• The quadratic equation.

$$ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

• e.g.,

$$2x^2 + 4x + 2 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(2)}}{2(2)} = -1 \pm \sqrt{0} = -1, -1$$

$$\therefore (x+1)(x+1) = 0$$

note the signs

• Complex roots will occur when,

$$b^2 - 4ac < 0$$

• Cubic equations can be solved explicitly, although this is too much for common memorization.

$$x^3 + ax^2 + bx + c = 0 = (x - r_1)(x - r_2)(x - r_3)$$

First, calculate,

$$Q = \frac{3b - a^2}{9} \qquad R = \frac{9ab - 27c - 2a^3}{54} \qquad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}} \qquad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then the roots,

$$r_1 = S + T - \frac{a}{3} \qquad r_2 = \frac{S+T}{2} - \frac{a}{3} + \frac{j\sqrt{3}}{2}(S - T) \qquad r_3 = \frac{S+T}{2} - \frac{a}{3} - \frac{j\sqrt{3}}{2}(S - T)$$

• On a few occasions a quartic (4th order) equation will also have to be solved. This can be done by first reducing the equation to a quadratic,

$$x^4 + ax^3 + bx^2 + cx + d = 0 = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

First, solve the equation below to get a real root (call it 'y'),

$$y^3 - by^2 + (ac - 4d)y + (4bd - c^2 - a^2 d) = 0$$

Next, find the roots of the 2 equations below,

$$r_1, r_2 = z^2 + \left(\frac{a + \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y + \sqrt{y^2 - 4d}}{2}\right) = 0$$

$$r_3, r_4 = z^2 + \left(\frac{a - \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y - \sqrt{y^2 - 4d}}{2}\right) = 0$$

• In Scilab,

```
x = poly(0, 'x');
roots(3 * x^2 + 4 * x + 2)

q = [2, 4, 3];
p = poly(q, 'x', 'coeff'); // defines a polynomial using a vector
roots(p)

derivative(p, 'x'); // finds the derivative of the polynomial
horner(p, 5); // evalautes the polynomial at 5
s = (x + 1) / p; // an algebra operation
```

29.2 Special Forms

29.2.1 Completing the Square

$$x^2 + Ax + B = (x + C)^2 + D$$

$$= x^2 + 2Cx + (C^2 + D)$$

$$A = 2C \qquad B = C^2 + D$$

$$C = \frac{A}{2} \qquad D = B - C^2$$

for example, given,

$$5x^2 + 50x + 10$$

$$= 5(x^2 + 10x + 2)$$

$$= 5((x + 5)^2 - 23)$$

29.2.2 Newton-Raphson to Find Roots

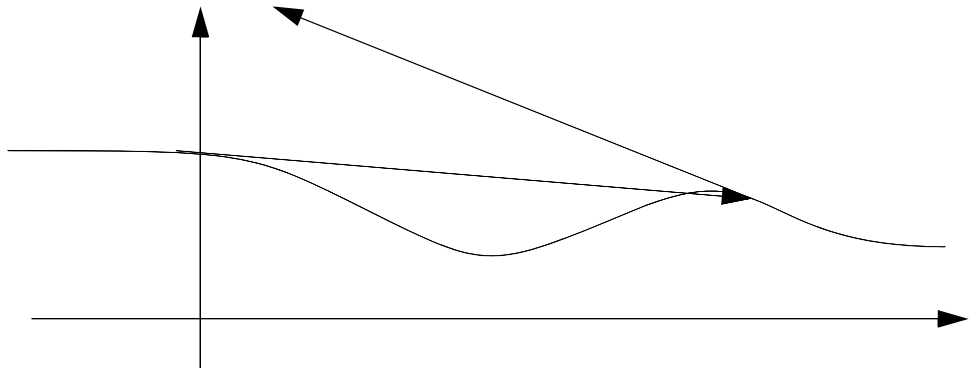
• When given an equation where an algebraic solution is not feasible, a numerical solution may be required. One simple technique uses an instantaneous slope of the function, and takes iterative steps towards a solution.

$$x_{i+1} = x_i - \frac{f(x_i)}{\left(\frac{d}{dx}f(x_i)\right)}$$

• The function f(x) is supplied by the user along with an initial guess.

• This method can become divergent if the function has an inflection point near the root.

• The technique is also sensitive to the initial guess.



• This calculation should be repeated until the final solution is found.

• Scilab example,

NEWTON RAPHSON ROOT EXAMPLE

29.3 Complex Numbers

• Complex values

The Complex (imaginary) Number:

$$j = \sqrt{-1} \qquad j^2 = -1$$

Complex Numbers:

$a + bj$ where, a and b are both real numbers

Complex Conjugates (denoted by adding an asterisk '*' to the variable):

$$N = a + bj \qquad N* = a - bj$$

Basic Properties:

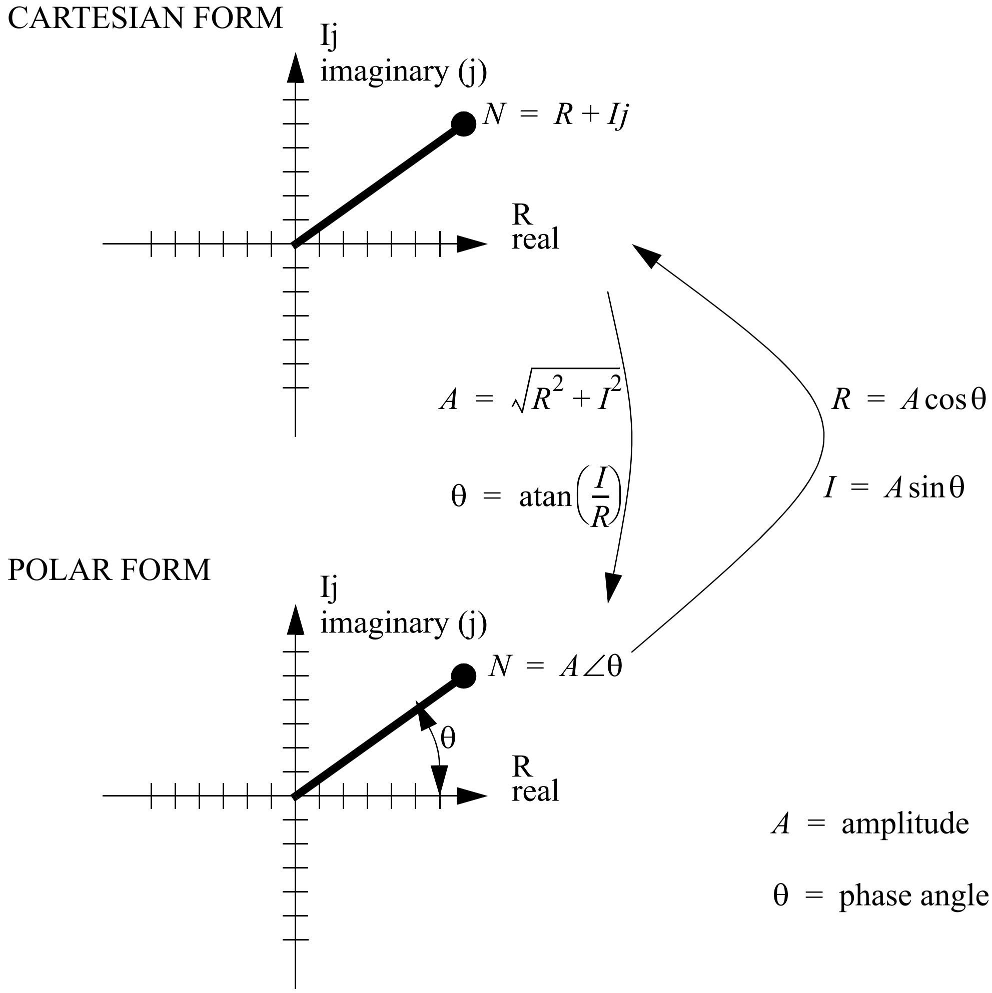
$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

$$(a + bj) \cdot (c + dj) = (ac - bd) + (ad + bc)j$$

$$\frac{N}{M} = \frac{a + bj}{c + dj} = \frac{N}{M}\left(\frac{N*}{N*}\right) = \left(\frac{a + bj}{c + dj}\right)\left(\frac{c - dj}{c - dj}\right) = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)j$$

- We can also show complex numbers graphically. These representations lead to alternative representations. If it is not obvious above, please consider the notation uses a cartesian notation, but a polar notation can also be very useful when doing large calculations.



• We can also do calculations using polar notation (this is well suited to multiplication and division, whereas cartesian notation is easier for addition and subtraction),

$$A\angle\theta = A(\cos\theta + j\sin\theta) = Ae^{j\theta}$$

$$e^{A+jB} = e^{A}e^{jB} = e^{A}(\cos\theta + j\sin\theta)$$

$$(A_1\angle\theta_1)(A_2\angle\theta_2) = (A_1A_2)\angle(\theta_1 + \theta_2)$$

$$\frac{(A_1\angle\theta_1)}{(A_2\angle\theta_2)} = \left(\frac{A_1}{A_2}\right)\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta)^n = (A^n)\angle(n\theta) \qquad \text{(DeMoivre's theorem)}$$

• Note that DeMoivre's theorem can be used to find exponents (including roots) of complex numbers

• Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\text{Note: for} \quad 0 + 1j = \cos\theta + j\sin\theta$$
$$\theta = \frac{\pi}{2}$$
$$e^{j\frac{\pi}{2}} = j$$

• From the above, the following useful identities arise:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

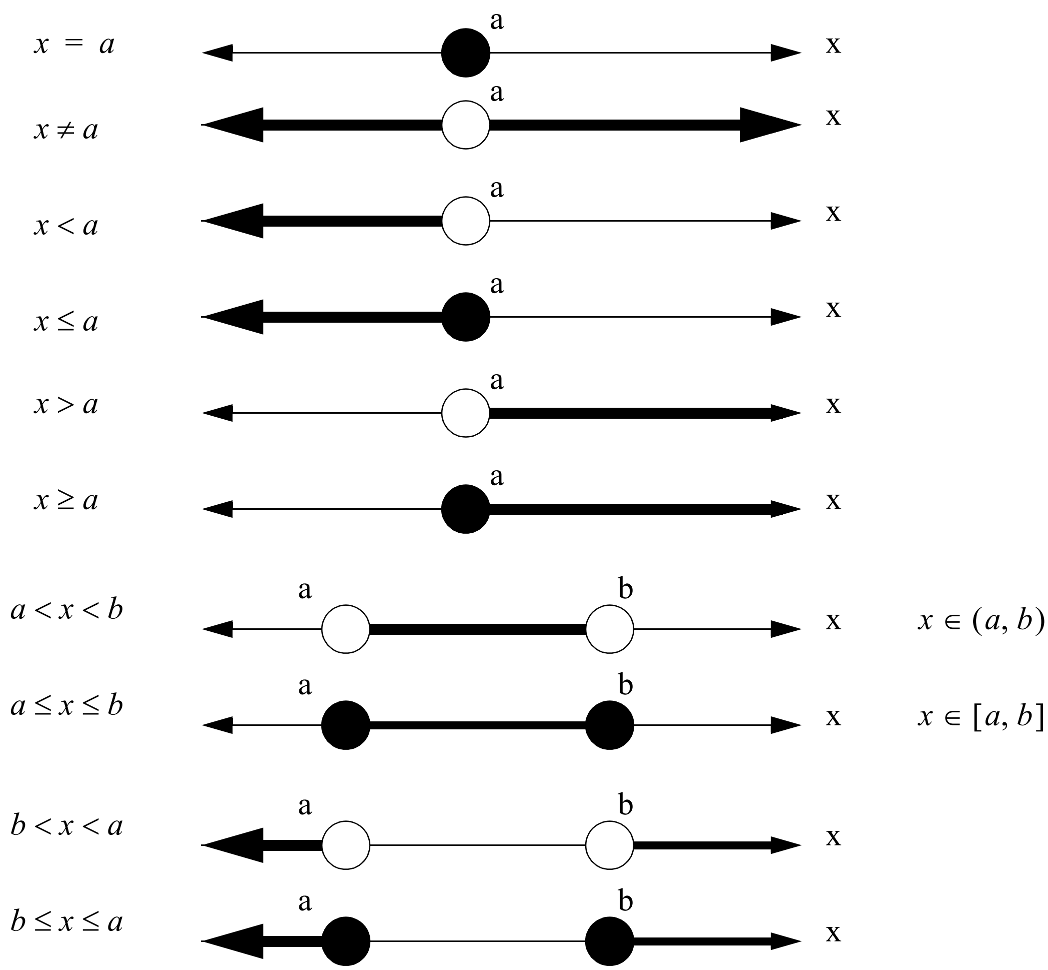
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

• In Scilab

```
j = sqrt(-1);
A = (1 + 2 * j) / ( 3 + 4 * j);
A
[mag, theta] = polar(A);
mag, theta // the magnitude and angle of the complex angle A
abs(A); // the magnitude of the complex number
real(A); // the real part of the complex number
imag(A); // the imaginary part of the complex number
conj(A); // the conjugate of the complex number
atan(imag(A), real(A)); // the angle of the complex number
```

29.4 Equality and Inequality

• Some basic relationships are illustrated below with number lines. The shaded dots indicate that the values include the point. Unshaded dots indicate that the values approach but do not equal the value.



- rearranging equations with inequalities

$$a + b < 5 \qquad \therefore a < 5 - b$$

- When doing calculations there is some roundoff error that result in a number that should be zero but has a finite value. Consider single precision floating point numbers with 7 digits, or double precision with 14, the last digit is equality in numerical calculations using a tolerance. In these cases a tolrance is used between the target and actual values to determine equality.

$$\varepsilon = |val| - \text{target}$$

$$if(\varepsilon < 0.000001)then\varepsilon = 0$$

• In Scilab,

```
if ( x > A ) | (x < B) then
        // program statement
else if x == A then
        // more program statements
end
```

• Another Scilab example,

$$\begin{aligned} F(x) &= 5 & & 0 < x \le 5 \\ &= 0 & & x \le 0, x > 5 \end{aligned}$$

```
function foo = F(x)
        if (x > 0) & (x <= 5) then
                        foo = 5;
        else
                        foo = 0;
        end
end_function
```

29.5 Functions

• Functions help to identify and organize self contained expressions.

• defining functions encourages reuse of mathematical expresions or program code

• terminology,

$f(x)$ (say 'f of x')

• an example function is,

$$f(x) = x^2 + 3$$

$$f(2) = 2^2 + 3 = 7$$

• In Scilab the function becomes,

```
function foo = f(x)
        foo = x^2 + 3;
endfunction

mprintf("%f \n", f(2));
```

Note: Use indents when writing programs to ensure structure. While these are not necessary they will make debugging much easier.

Note: Using functions is not necessary in simpler programs, but larger programs will become very difficult to work with if functions are not used.

29.6 Special Functions

29.6.1 Logarithms

• Logarithms also have a few basic properties of use,

The basic base 10 logarithm:

$$\log x = y \qquad\qquad x = 10^y$$

The basic base n logarithm:

$$\log_n x = y \qquad\qquad x = n^y$$

The basic natural logarithm (e is a constant with a value found near the start of this section:

$$\ln x = \log_e x = y \qquad\qquad x = e^y$$

• All logarithms observe a basic set of rules for their application,

$$\log_n(xy) = \log_n(x) + \log_n(y)$$

$$\log_n(n) = 1$$

$$\log_n(1) = 0$$

$$\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$$

$$\log_n(x^y) = y\log_n(x)$$

$$\log_n(x) = \frac{\log_m(x)}{\log_m(n)}$$

$$\ln(A\angle\theta) = \ln(A) + (\theta + 2\pi k)j \qquad k \in I$$

e.g., solve

$$5 = 10^n$$

$$\log(5) = n\log(10)$$

$$n = \frac{\log(5)}{\log(10)}$$

• Note: most computers use the 'log' function for the natural logarithm. The base 10 logarithm is normally 'log10'.

• In Scilab this is

```
log10(3) - base 10 log
log(3) - natural log
log(exp(10))
```

29.7 Solving Systems of Linear Equations

• Systems of linear equations are of the form below,

$$x + y = 5$$

$$2x + 3y = 8$$

• In general there must be the same number of equations and unknowns to solve the equations.
• In some cases the equations will not be solvable. In this case we say the equations are singular.

29.7.1 Substitution

• Substitution is the most fundamental method for solving linear equations, but it is the least routine,

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

To solve for y, we substitute to eliminate x.

$$\therefore x = 5 - y$$

$$\therefore 2(5 - y) + 3y = 8$$

$$\therefore y = -2$$

To solve for x, we substitute the value for y into an earlier equation.

$$\therefore x = 5 - (-2) = 7$$

• In Scilab this is

EXAMPLE ????????????

29.7.2 Addition

• Polynomials can be added to eliminate variables (in a matrix for this is referred to as the Gauss-Jordan row reduction method).

Given,

$$x + y = 5$$
$$2x + 3y = 8$$

multiply and add the equations,

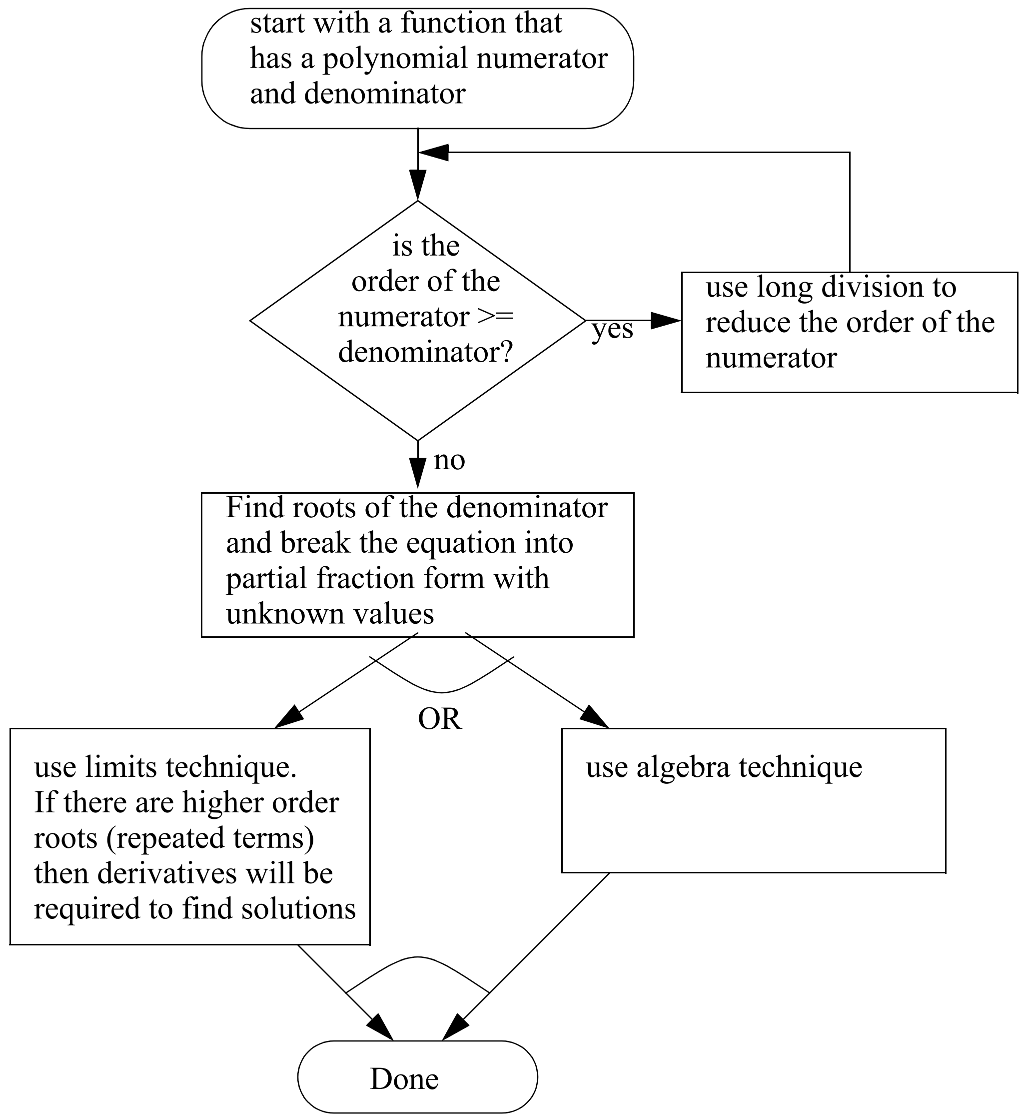
$$\begin{array}{r} (-2)(x + y = 5) \\ 2x + 3y = 8 \\ \hline 0x + 1y = -2 \end{array} \qquad \therefore y = -2$$

$$x = 5 - (-2) = 7$$

29.8 Simplifying Polynomial Expressions

29.8.1 Partial Fractions

• The next is a flowchart for partial fraction expansions.



• The partial fraction expansion for,

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}[-(s+1)^{-2}] = -1$$

- Consider the example below where the order of the numerator is larger than the denominator.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r|l} & 5s + 3 \\ \hline s^2 + 4 & 5s^3 + 3s^2 + 8s + 6 \\ & \underline{5s^3 + 20s} \\ & \quad 3s^2 - 12s + 6 \\ & \quad \underline{3s^2 + 12} \\ & \qquad -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

- When the order of the denominator terms is greater than 1 it requires an expanded partial fraction form, as shown below.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

• We can solve the previous problem using the algebra technique.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

29.8.2 Summation and Series

• The notation $\sum_{i=a}^{b} x_i$ is equivalent to $x_a + x_{a+1} + x_{a+2} + \ldots + x_b$ assuming a and b are integers and $b \geq a$. The index variable i is an index often replaced with j, k, m, and n.

• Operations on summations:

$$\sum_{i=a}^{b} x_i = \sum_{i=b}^{a} x_i$$

$$\sum_{i=a}^{b} \alpha x_i = \alpha \sum_{i=a}^{b} x_i$$

$$\sum_{i=a}^{b} x_i + \sum_{j=a}^{b} y_j = \sum_{i=a}^{b} (x_i + y_i)$$

$$\sum_{i=a}^{b} x_i + \sum_{i=b+1}^{c} x_i = \sum_{i=a}^{c} x_i$$

$$\left(\sum_{i=a}^{b} x_i\right)\left(\sum_{j=c}^{d} y_j\right) = \sum_{i=a}^{b} \sum_{j=c}^{d} x_i y_j$$

• Some common summations:

$$\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$$

$$\sum_{i=0}^{N-1} \alpha^i = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N, & \alpha = 1 \end{cases}$$ for both real and complex α.

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, \ |\alpha| < 1$$ for both real and complex α. For $|\alpha| \geq 1$, the summation does not converge.

• In Scilab

LOOP EXAMPLE

29.9 Limits

• Limits stuff goes here.....

29.10 Problems

1. Rearrange the following equation so that only ‘y’ is on the left hand side.

$$\frac{y+x}{y+z} = x+2$$ ans. $$y = \frac{-x+xz+2z}{-x-1}$$

2. Solve the following equation to find ‘x’.

$$2x^2 + 8x = -8$$ ans. x= -2, -2

3. Solve the following system of equations using substitution.

$$x + 2y + 3z = 5$$
$$x + 4y + 8z = 0$$
$$4x + 2y + z = 1$$

ans. x=-7. y=18.75,z=-8.5

4. Simplify the following expressions.

a) $n\log(x) + m\log(y) - \log(z)$

ans. $\log\left(\frac{x^n y^m}{z}\right)$

5. Simplify,

	ans.
$(5x + 3y) - (2x - 7y)$	$3x + 10y$
$(5x^2 + 3y) - (2x - 7y)$	$5x^2 - 2x + 4y$
$\frac{x^2 + 5x + 6}{x^2 + 7x + 12}$	$\frac{x + 2}{x + 4}$
$\frac{5}{x} + \frac{7}{3x}$	$\frac{22}{3x}$
$\frac{5}{x} - \frac{7}{3x}$	$\frac{8}{3x}$
$\frac{5}{3 - \frac{7}{x}}$	$\frac{5x}{3x - 7}$
$5(x + 3) - 9(x - 6)$	$-4x + 69$
$\frac{5}{x - 2} = \frac{6}{x + 3}$	$x = 27$

6. Multiply,

	ans.
$(2x + 3)(5x + 4)$	$10x^2 + 23x + 12$
$(2x^2 + 3x + 4)(5x + 4)$	$10x^3 + 23x^2 + 32x + 16$

7. Factor,

$x^2 - 4$	ans. $(x+2)(x-2)$
$x^2 - 4y^2$	$(x+2y)(x-2y)$
$x^2 - 4x + 4$	$(x-2)(x-2)$
$x^2 + 5x + 6$	$(x+2)(x+3)$

10. solve for x, y,

$$5x + 3y = 10$$
$$x - y = 0$$

ans. x=y=1.25

11. Find roots,

$$x^2 + 2x + 100 = 0$$
$$x^4 + 2x^2 + 100 = 0$$

ans.
$$-1 \pm \frac{\sqrt{-396}}{4}$$
$$x = \pm\sqrt{-1 \pm \frac{\sqrt{-396}}{4}}$$

13. Convert the following log to base 10,

$\log_6 3$ ans. 0.613

14. Find x,

$x^{12} = 1372$ ans. x = 1.826

$$5^{12+x} = 1372$$

15. Simplify,

$$\frac{\log A + \log B}{C \log D}$$ ans. $$\frac{1}{C}\log(AB - D)$$

16. Find the polar form for,

$$5 + 6j$$

ans. $7.81\angle 0.876rad$

17. Find the cartesian complex value for,

$$10\angle 0.13rad$$

ans. $9.92 + 1.30j$

18. Find the cartesian complex value for,

$$(10\angle 0.13rad)^3$$

ans. $924.9 + 380.2j$

19. Solve,

$$-5x - 2 > 3x + 4$$

$$x^2 - 36 > 0$$

$$x^2 + 36 > -12x$$

ans.

$x < -0.75$

$|x| > 6$

$x >$

20. Solve,

$$(8 - 3j) - (12 + 4j)$$

$$(4 + 5j)^2$$

$$\left(\frac{8 - 3j}{4 + 5j}\right)$$

$$(x - 6 - 4j)(x - 6 + 4j)$$

ans.

$-4 - 7j$

$-9 - 40j$

$0.415 - 1.268j$

$x^2 - 12x + 52$

21. Find the limits below.

a) $$\lim_{t \to 0}\left(\frac{t^3 + 5}{5t^3 + 1}\right)$$ ans. 5

b) $$\lim_{t \to \infty}\left(\frac{t^3 + 5}{5t^3 + 1}\right)$$ ans. 1/5

22. Evaluate the limits,

$$\lim_{t \to \infty} \frac{\sin(t)}{t}$$

$$\lim_{t \to 0} \frac{\sin(t)}{t}$$

$$\lim_{t \to \infty} \sin(t)$$

$$\lim_{t \to 0} \sin(t)$$

$$\lim_{t \to \infty} te^{-t}$$

$$\lim_{t \to 0} te^{-t}$$

ans.

23. Reduce the following expression to partial fraction form.

$$\frac{x^2 + 4}{x^3 + 6x^2 + 9x}$$

ans. $$\frac{4}{9x} - \frac{13}{3(x+3)^2} + \frac{5}{9(x+3)}$$

29.11 Challenge Problems

1. Write a program (in Scilab, C, etc.) that will perform the following calculation.

$$y(x) = \sum_{i=1}^{n} (x+i)^2$$

2. Develop a program to fit a polynomial to set set of points. In the case where the order of the polynomial does not allow an exact fit, a least squares method should be used to obtain the best fit.

3. Write a program in Scilab (ask if you would prefer to use another platform). The program should use the Newton-Raphson method to find the zeros of an arbitrary function. The functions below should be used for testing the program. The final program will be tested for robustness. The final program should be structured.

$$x^2 + x + 5 = 0$$

$$-5x^3 + x^2 + 10 = 0$$

30. TRIGONOMETRY

Topics:

-

Objectives:

-

30.1 Introduction

• Angles degrees and radians

$$360° = 2\pi radians$$

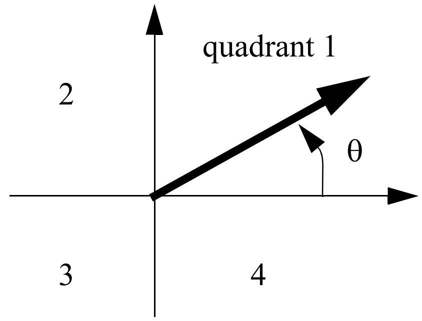
$$380° = 20° = -340°$$

$$60\text{minutes} = 60' = 1°$$

$$60\text{seconds} = 60'' = 1'$$

• Most computers do calculations in radians

• Angle quadrants,



30.1.1 Functions

• The basic trigonometry functions are,

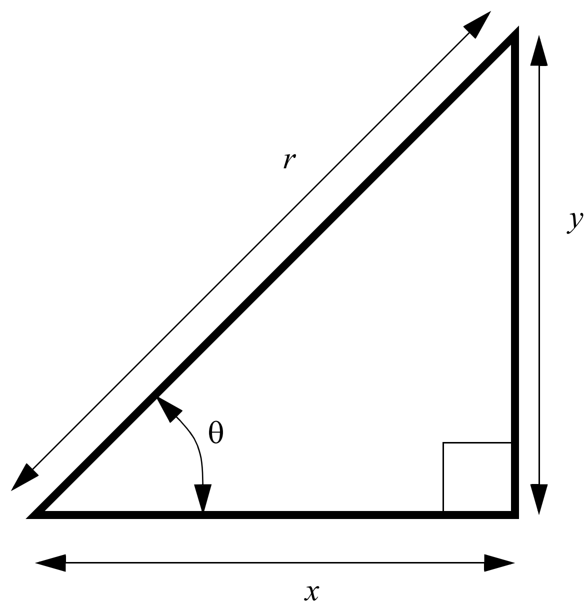
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

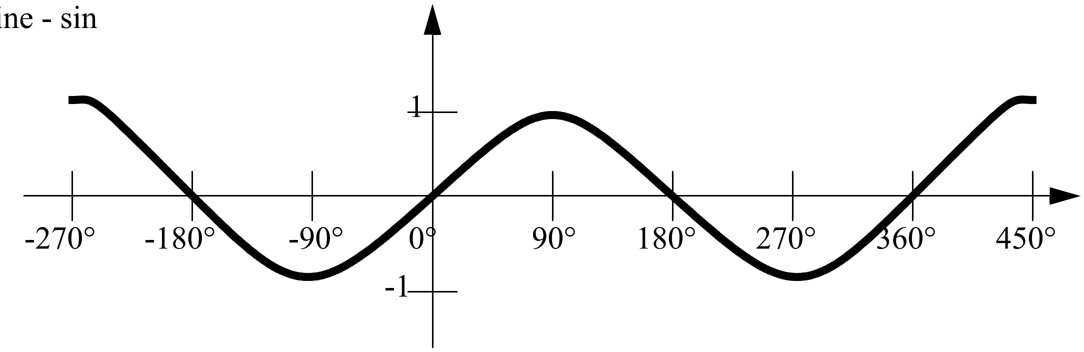
$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

Pythagorean Formula:

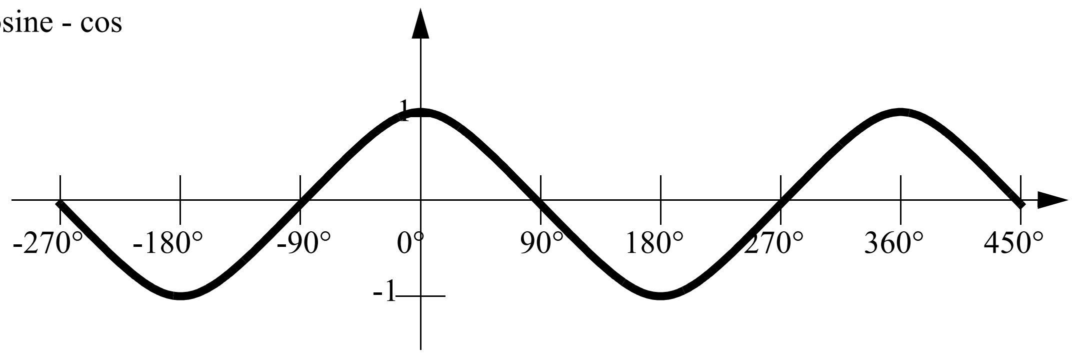
$$r^2 = x^2 + y^2$$



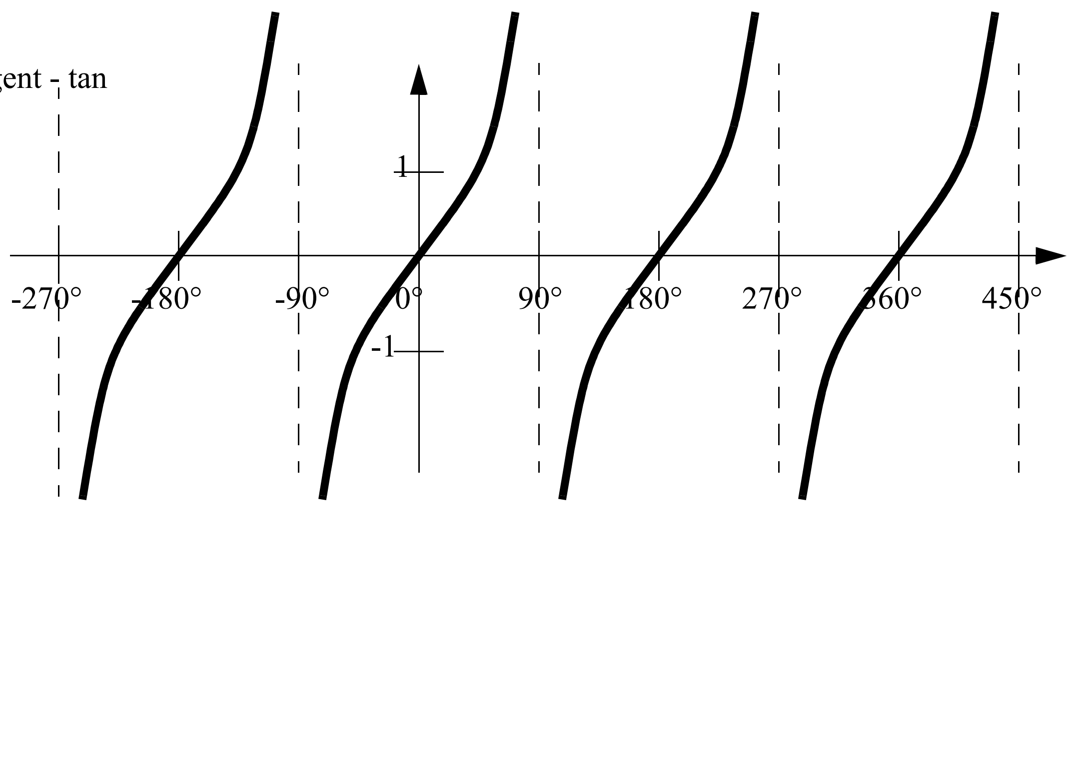
• Graphs of these functions are given below, Sine - sin

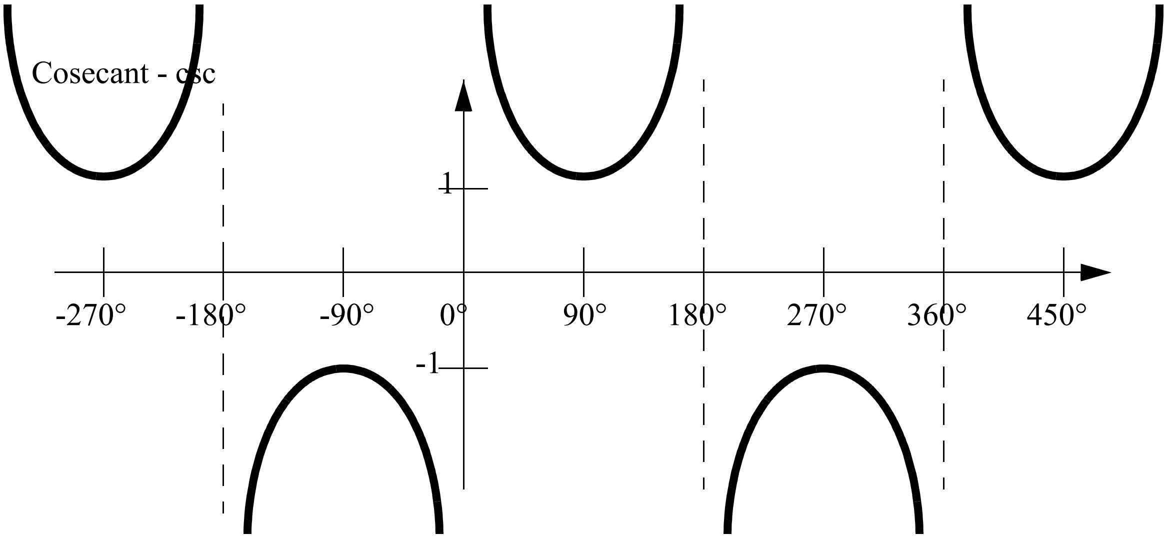


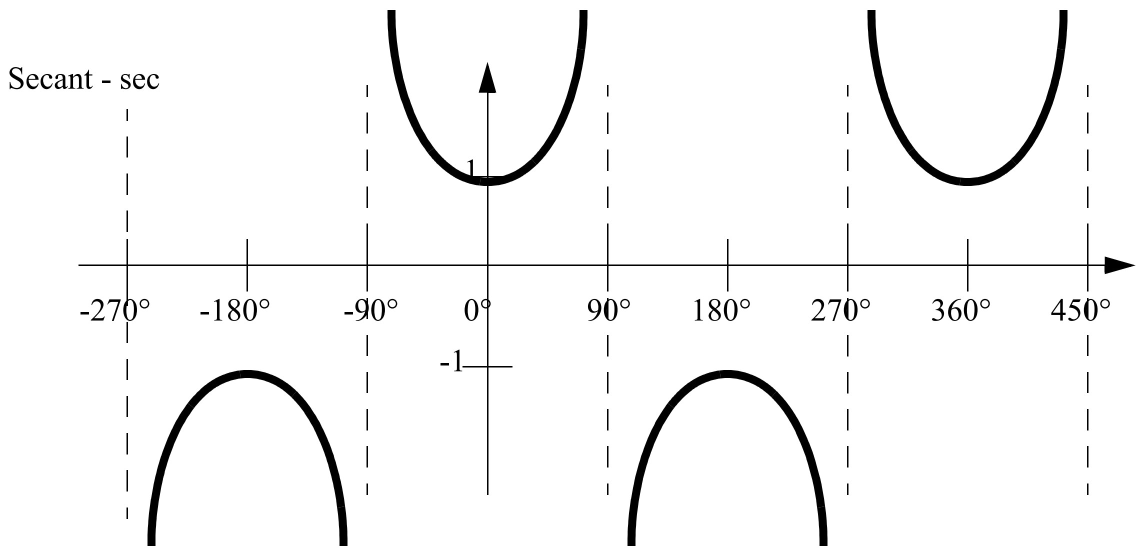
Cosine - cos

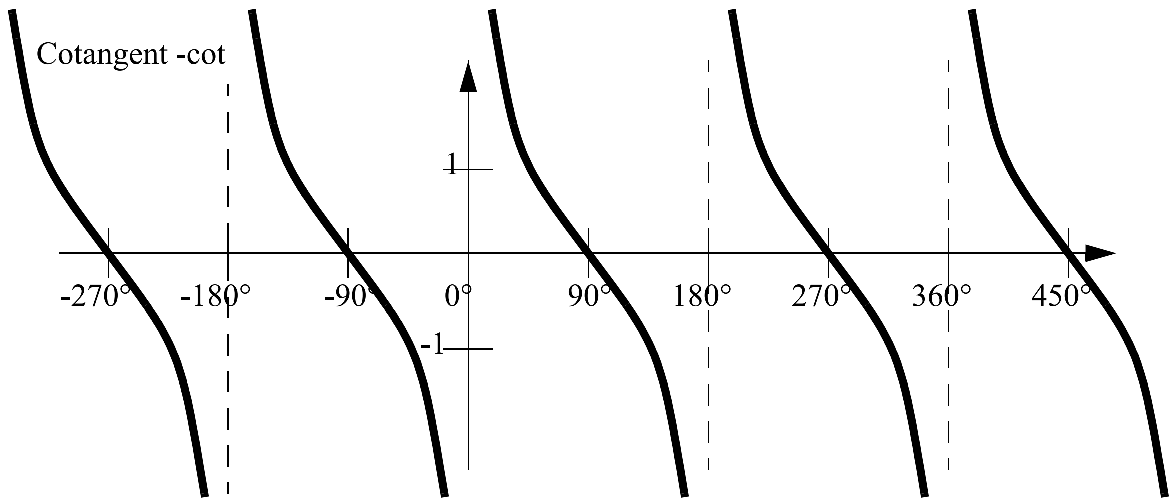


Tangent - tan




Cosecant - csc
1
-1
-270°
-180°
-90°
0°
90°
180°
270°
360°
450°


Secant - sec
1
-1
-270°
-180°
-90°
0°
90°
180°
270°
360°
450°



30.1.2 Inverse Functions

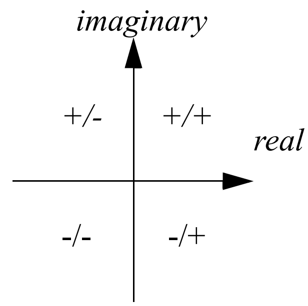
• Inverse Functions

$$\tan^{-1}\left(\frac{y}{x}\right) = \operatorname{atan}\left(\frac{y}{x}\right) = \theta$$

$$\sin^{-1}\left(\frac{y}{r}\right) = \operatorname{asin}\left(\frac{y}{r}\right) = \theta$$

$$\cos^{-1}\left(\frac{x}{r}\right) = \operatorname{acos}\left(\frac{x}{r}\right) = \theta$$

Note: recall that $\tan\theta = \frac{Re}{Im}$ *but the* $\operatorname{atan}\theta$ *function in calculators and software only returns values between -90 to 90 degrees. To compensate for this the sign of the real and imaginary components must be considered to determine where the angle lies. If it lies beyond the -90 to 90 degree range the correct angle can be obtained by adding or subtracting 180 degrees.*



• Note: trig calculations can take a while and should be minimized or avoided in programs.

• Scilab example,

```
sin(3.14159)
asin(0.5)
cos(3.14159)
acos(0.5)
tan(3.14159)
atan(0.5)
atan(1.0, 0.5)
```

30.1.3 Triangles

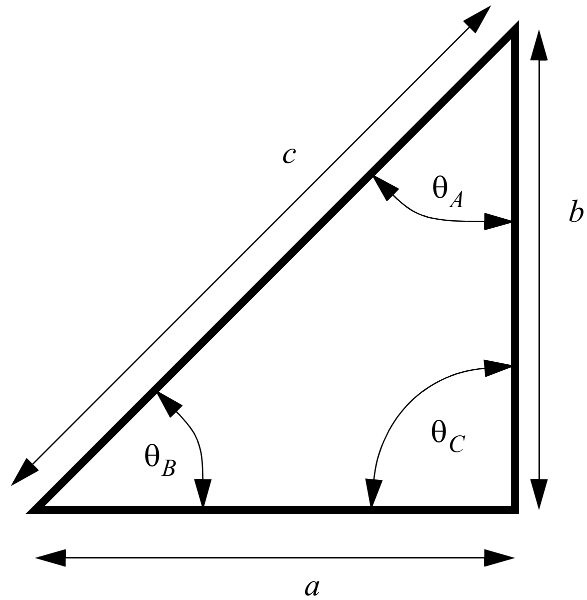
• NOTE: Keep in mind when finding these trig values, that any value that does not lie in the right hand quadrants of cartesian space, may need additions of ±90° or ±180°.

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$

Sine Law:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}$$



30.1.4 Relationships

• Now a group of trigonometric relationships will be given. These are often best used when attempting to manipulate equations.

$$\sin(\theta) = \sin(\theta \pm 360n) \qquad n \in I$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\theta = \cos(\theta - 90°) = \cos(90° - \theta) = \text{etc.}$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \qquad \text{OR} \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \qquad \text{OR} \qquad \cos(2\theta) = (\cos\theta)^2 - (\sin\theta)^2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1\tan\theta_2} \qquad 1 + (\tan\theta)^2 = (\sec\theta)^2$$

$$\cot(\theta_1 \pm \theta_2) = \frac{\cot\theta_1\cot\theta_2 \mp 1}{\tan\theta_2 \pm \tan\theta_1} \qquad 1 + (\cot\theta)^2 = (\csc\theta)^2$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

-ve if in left hand quadrants

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

• Scilab for trig identities,

EXAMPLES OF TRIG IDENTITIES

• These can also be related to complex exponents,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

30.1.5 Hyperbolic Functions

• The basic definitions are given below,

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} = \text{hyperbolic sine of x}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = \text{hyperbolic cosine of x}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \text{hyperbolic tangent of x}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}} = \text{hyperbolic cosecant of x}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}} = \text{hyperbolic secant of x}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \text{hyperbolic cotangent of x}$$

• some of the basic relationships are,

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x)$$
$$\tanh(-x) = -\tanh(x)$$
$$\operatorname{csch}(-x) = -\operatorname{csch}(x)$$
$$\operatorname{sech}(-x) = \operatorname{sech}(x)$$
$$\coth(-x) = -\coth(x)$$

• Some of the more advanced relationships are,

$$(\cosh x)^2 - (\sinh x)^2 = (\operatorname{sech} x)^2 + (\tanh x)^2 = (\coth x)^2 - (\operatorname{csch} x)^2 = 1$$
$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$
$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$
$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

• Some of the relationships between the hyperbolic, and normal trigonometry functions are,

$$\sin(jx) = j\sinh(x) \qquad j\sin(x) = \sinh(jx)$$
$$\cos(jx) = \cosh(x) \qquad \cos(x) = \cosh(jx)$$
$$\tan(jx) = j\tanh(x) \qquad j\tan(x) = \tanh(jx)$$

30.1.6 Special Relationships

• The Small Angle Approximation

$$\sin\theta = \theta \qquad \text{when} \qquad \theta \approx 0$$

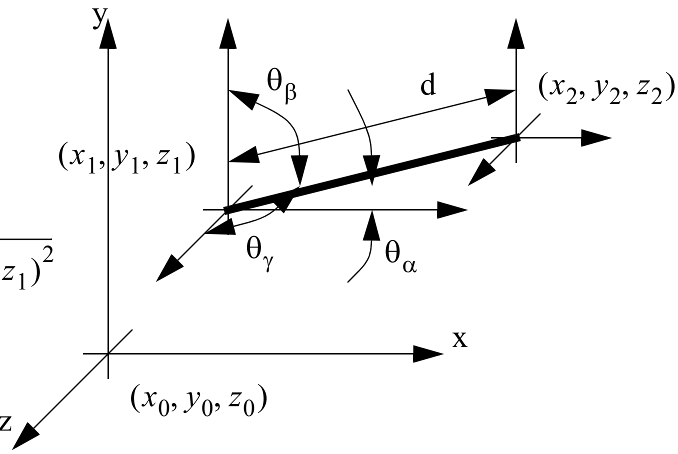
30.1.7 Planes, Lines, etc.

• The most fundamental mathematical geometry is a line. The basic relationships are given below,

$y = mx + b$	defined with a slope and intercept
$m_{\text{perpendicular}} = \frac{1}{m}$	a slope perpendicular to a line
$m = \frac{y_2 - y_1}{x_2 - x_1}$	the slope using two points
$\frac{x}{a} + \frac{y}{b} = 1$	as defined by two intercepts

• If we assume a line is between two points in space, and that at one end we have a local reference frame, there are some basic relationships that can be derived.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



The direction cosines of the angles are,

$$\theta_\alpha = \operatorname{acos}\left(\frac{x_2 - x_1}{d}\right) \qquad \theta_\beta = \operatorname{acos}\left(\frac{y_2 - y_1}{d}\right) \qquad \theta_\gamma = \operatorname{acos}\left(\frac{z_2 - z_1}{d}\right)$$

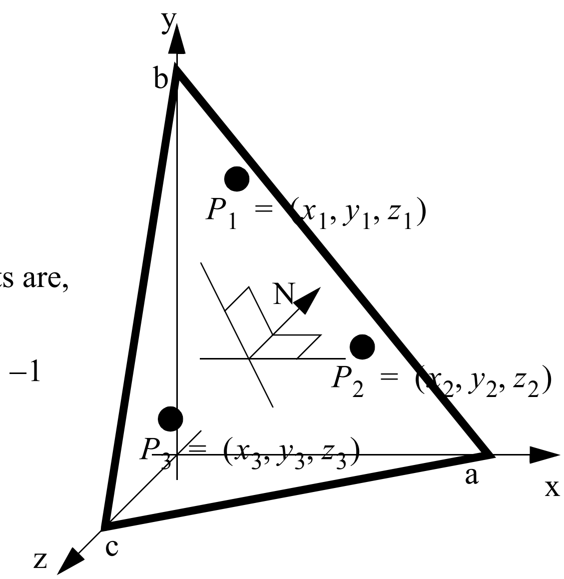
$$(\cos\theta_\alpha)^2 + (\cos\theta_\beta)^2 + (\cos\theta_\gamma)^2 = 1$$

The equation of the line is,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \text{Explicit}$$

$$(x, y, z) = (x_1, y_1, z_1) + t((x_2, y_2, z_2) - (x_1, y_1, z_1)) \qquad \text{Parametric t=[0,1]}$$

• The relationships for a plane are,



The explicit equation for a plane is,

$$Ax + By + Cy + D = 0$$

where the coefficients defined by the intercepts are,

$$A = \frac{1}{a} \qquad B = \frac{1}{b} \qquad C = \frac{1}{c} \qquad D = -1$$

The determinant can also be used,

$$det\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{bmatrix} = 0$$

$$\therefore det\begin{bmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{bmatrix}(x-x_1) + det\begin{bmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{bmatrix}(y-y_1)$$

$$+ det\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}(z-z_1) = 0$$

The normal to the plane (through the origin) is,

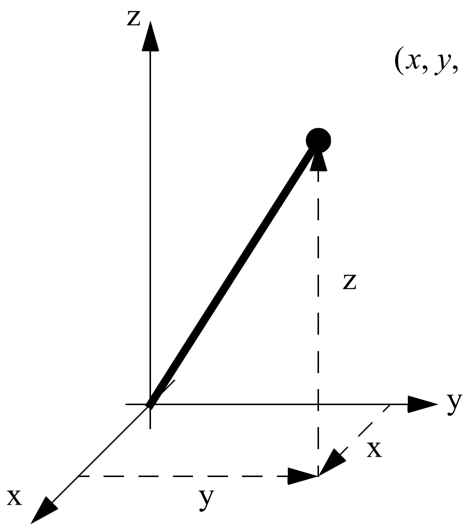
$$(x, y, z) = t(A, B, C)$$

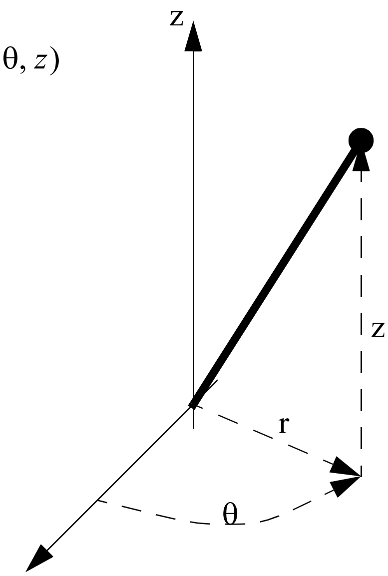
30.2 Coordinate Systems

30.2.1 Cylindrical Coordinates

• Basically, these coordinates appear as if the cartesian box has been replaced with a cylinder,

$$(x, y, z) \leftrightarrow (r, \theta, z)$$





$$x = r\cos\theta$$

$$y = r\sin\theta$$

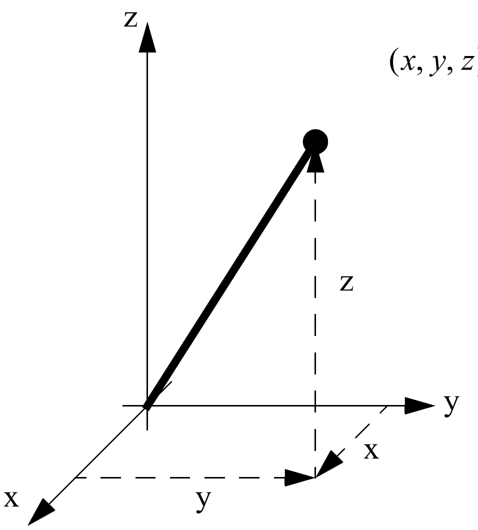
$$r = \sqrt{x^2 + y^2}$$

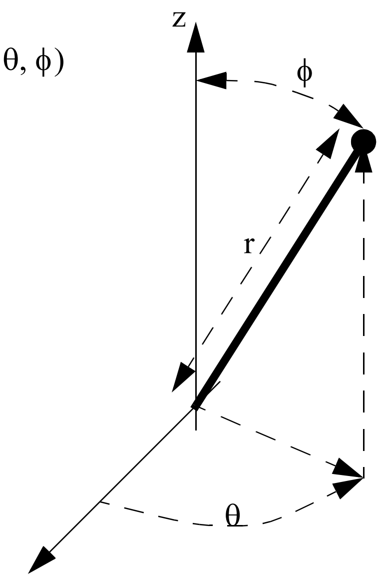
$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

30.2.2 Spherical Coordinates

• This system replaces the cartesian box with a sphere,

$(x, y, z) \leftrightarrow (r, \theta, \phi)$





$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

$$\phi = \operatorname{acos}\left(\frac{z}{r}\right)$$

30.3 Problems

1. For the following angles, i) indicate the quadrants, ii) write the sine, cosine, and tangent values, iii) write an expression for all equivalent angles.

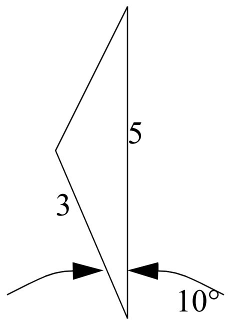
a) $20°$

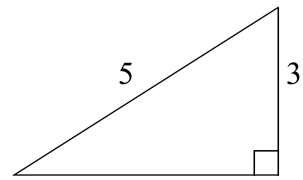
b) $2rad$

c) $-2rad$

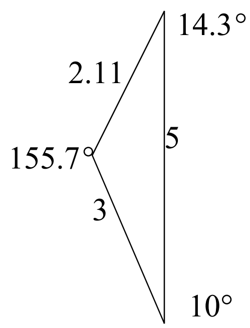
d) $2000°$

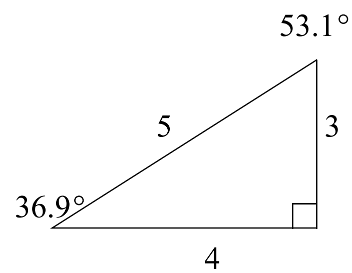
2. Find all of the missing side lengths and corner angles on the two triangles below,





ans.





3. A line that passes through the point (1, 2) and has a slope of 2. Find the equation for the line, and for a line perpendicular to it passing through the given point. (ans. y = 2x + a, y = -0.5x+b)

4. Convert the following angles to/from degrees or radians.

to rad.: 30°, 280°, 192°5'30"

to deg.: $\frac{5\pi}{36}rad, \frac{7\pi}{12}rad$

ans. $\frac{\pi}{6}rad, \frac{14}{9}\pi rad, 1.067\pi rad$

25°, 105°

5. On a circle with a diameter of 0.5in. what is,

a) the arc length for a 1 rad angle

b) the arc length for a 20 degree angle
c) the circumference.
d) the angle resulting in a 0.5m arc.

(ans. 0.25in, PI/36 in, 0.5 PI in., 78.7rad.)

6. If a 30cm radius rotating mass has a rotational rate of 200rpm, how fast is a point at the outside moving? What angular velocity is required for an outside speed of 2m/s? (ans. 6.28m/s, 6.67 rad/s)

7.Given the (x, y) points below, find the angle

a)	$(3, 4)$	(ans. $53°$
b)	$(-3, 4)$	$126°$
c)	$(3, -4)$	$-53°$
d)	$(-3, -4)$	$-126°$

8.Convert the following to a function with the smallest positive angle possible

a)	$\sin(200°)$	(ans. $-\sin(20°)$
b)	$\tan(-170°)$	$\tan(10°)$
c)	$\cos(325°)$	$\cos(35°)$
d)	$\cos(3.5\pi rad)$	$\cos(90°)$ or $\sin(0°)$

9.Given a triangle ABC find the missing side lengths or angles

θ_A	θ_A	θ_A	L_A	L_B	L_C

10. Simplify the following expressions.

$$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right)$$ (ans. = 1)

11.Prove the following

a) $\frac{\sin\theta\cos\theta}{\tan\theta} = \frac{(\cos\theta)^2 - (\sin\theta)^2}{1 - (\tan\theta)^2}$

b) $\sin\theta = 1 - \frac{(\cos\theta)^2}{1 + (\sin\theta)}$

c) $(1 + (\tan\theta)^2)(1 - (\sin\theta)^2) = 1$

d) $\sin(\theta_1 + \theta_2) - \sin(\theta_1 - \theta_2) = 2\cos\theta_1\sin\theta_2$

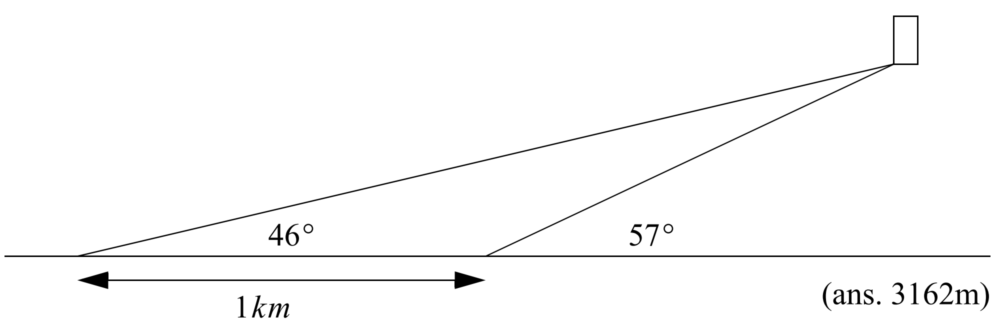
e) $\sin(\theta_1 + \theta_2)(-\sin(\theta_1 - \theta_2)) = (\sin(\theta_1))^2 - (\sin(\theta_2))^2$

f) $\tan\theta\sin 2\theta = 2(\sin\theta)^2$

12. Solve the equation,

$$(\cos\theta)^2 + 2\sin\theta - 2 = 0$$ (ans. $\theta =$

13. Two observers measure the height of a rocket by angle. Calculate the height of the rocket.



(ans. 3162m)

31. VECTORS

Topics:

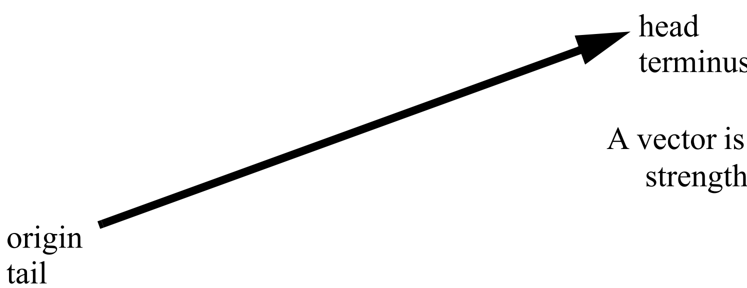
•

Objectives:

•

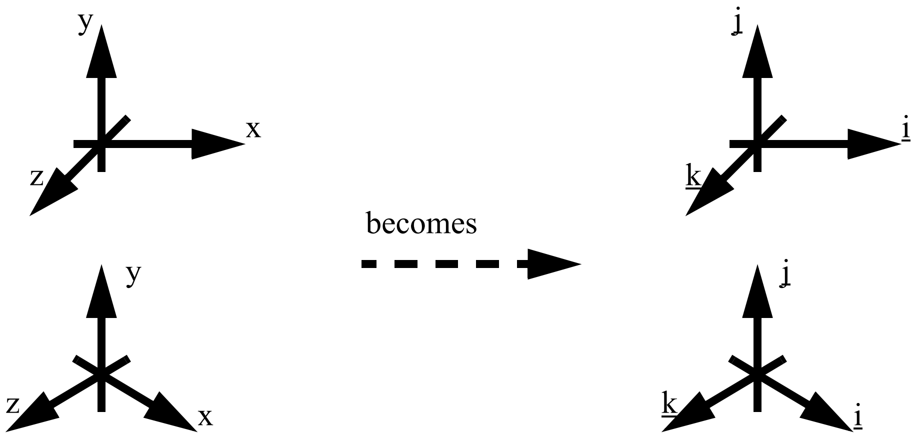
31.1 Introduction

• Vectors are often drawn with arrows, as shown below,



A vector is said to have magnitude (length or strength) and direction.

• Cartesian notation is also a common form of usage.

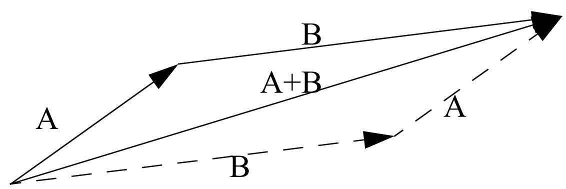


• Vectors can be added and subtracted, numerically and graphically,

$$A = (2, 3, 4) \qquad A + B = (2 + 7, 3 + 8, 4 + 9)$$

$$B = (7, 8, 9) \qquad A - B = (2 - 7, 3 - 8, 4 - 9)$$

Parallelogram Law



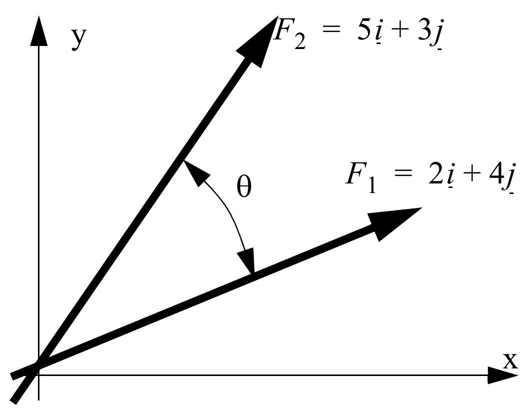
31.1.1 Dot (Scalar) Product

• We can use a dot product to find the angle between two vectors

$$\cos\theta = \frac{F_1 \bullet F_2}{|F_1||F_2|}$$

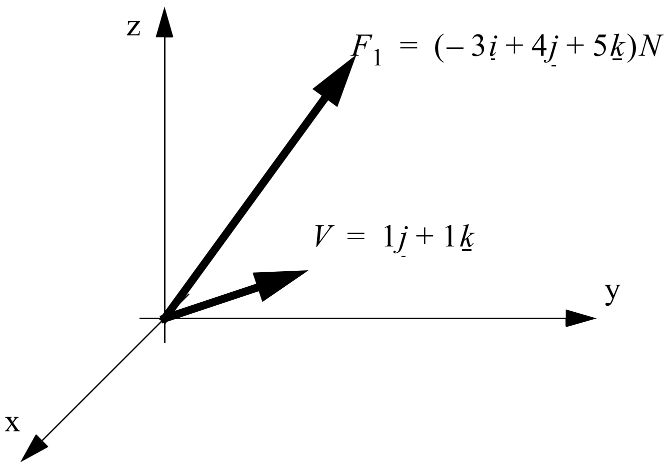
$$\therefore \theta = \operatorname{acos}\left(\frac{(2)(5) + (4)(3)}{\sqrt{2^2 + 4^2}\sqrt{5^2 + 3^2}}\right)$$

$$\therefore \theta = \operatorname{acos}\left(\frac{22}{(4.47)(6)}\right) = 32.5°$$



• We can use a dot product to project one vector onto another vector.

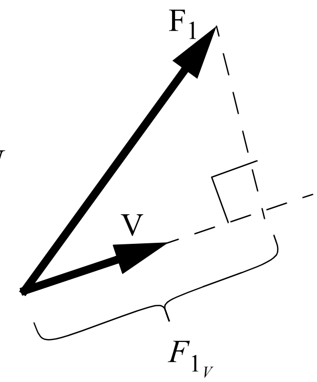
We want to find the component of force F_1 that projects onto the vector V. To do this we first convert V to a unit vector, if we do not, the component we find will be multiplied by the magnitude of V.



$$\lambda_V = \frac{V}{|V|} = \frac{1\underline{j} + 1\underline{k}}{\sqrt{1^2 + 1^2}} = 0.707\underline{j} + 0.707\underline{k}$$

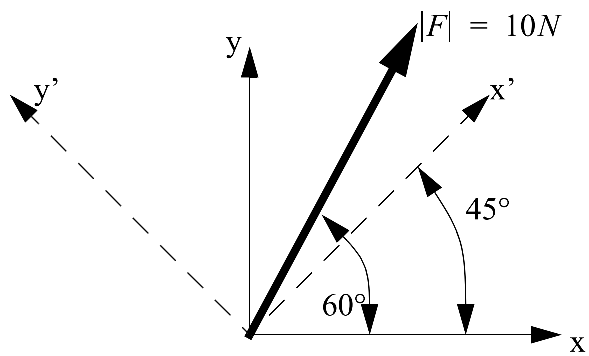
$$F_{1_V} = \lambda_V \bullet F_1 = (0.707\underline{j} + 0.707\underline{k}) \bullet (-3\underline{i} + 4\underline{j} + 5\underline{k})N$$

$$\therefore F_{1_V} = (0)(-3) + (0.707)(4) + (0.707)(5) = 6N$$



• We can consider the basic properties of the dot product and units vectors.

Unit vectors are useful when breaking up vector magnitudes and direction. As an example consider the vector, and the displaced x-y axes shown below as x'-y'.



We could write out 5 vectors here, relative to the x-y axis,

$$\text{x axis} = 2\underset{\sim}{i}$$
$$\text{y axis} = 3\underset{\sim}{j}$$
$$\text{x' axis} = 1\underset{\sim}{i} + 1\underset{\sim}{j}$$
$$\text{y' axis} = -1\underset{\sim}{i} + 1\underset{\sim}{j}$$
$$F = 10N\angle 60° = (10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}$$

None of these vectors has a magnitude of 1, and hence they are not unit vectors. But, if we find the equivalent vectors with a magnitude of one we can simplify many tasks. In particular if we want to find the x and y components of F relative to the x-y axis we can use the dot product.

$$\lambda_x = 1\underset{\sim}{i} + 0\underset{\sim}{j} \quad \text{(unit vector for the x-axis)}$$

$$F_x = \lambda_x \bullet F = (1\underset{\sim}{i} + 0\underset{\sim}{j}) \bullet [(10\cos 60°)\underset{\sim}{i} + (10\sin 60°)\underset{\sim}{j}]$$
$$\therefore \quad = (1)(10\cos 60°) + (0)(10\sin 60°) = 10N\cos 60°$$

This result is obvious, but consider the other obvious case where we want to project a vector onto itself,

$$\lambda_F = \frac{F}{|F|} = \frac{10\cos 60° i + 10\sin 60° j}{10} = \cos 60° i + \sin 60° j$$

Incorrect - Not using a unit vector

$$F_F = F \bullet F$$
$$= ((10\cos 60°)i + (10\sin 60°)j) \bullet ((10\cos 60°)i + (10\sin 60°)j)$$
$$= (10\cos 60°)(10\cos 60°) + (10\sin 60°)(10\sin 60°)$$
$$= 100((\cos 60°)^2 + (\sin 60°)^2) = \cancel{100}$$

Using a unit vector

$$F_F = F \bullet \lambda_F$$
$$= ((10\cos 60°)i + (10\sin 60°)j) \bullet ((\cos 60°)i + (\sin 60°)j)$$
$$= (10\cos 60°)(\cos 60°) + (10\sin 60°)(\sin 60°)$$
$$= 10((\cos 60°)^2 + (\sin 60°)^2) = 10$$

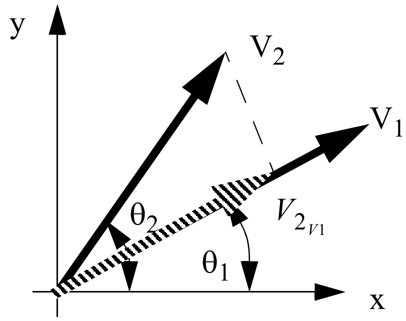
Correct

Now consider the case where we find the component of F in the x' direction. Again, this can be done using the dot product to project F onto a unit vector.

$$u_{x'} = \cos 45° i + \sin 45° j$$

$$F_{x'} = F \bullet \lambda_{x'} = ((10\cos 60°)i + (10\sin 60°)j) \bullet ((\cos 45°)i + (\sin 45°)j)$$
$$= (10\cos 60°)(\cos 45°) + (10\sin 60°)(\sin 45°)$$
$$= 10(\cos 60° \cos 45° + \sin 60° \sin 45°) = 10(\cos(60° - 45°))$$

Here we see a few cases where the dot product has been applied to find the vector projected onto a unit vector. Now finally consider the more general case,



First, by inspection, we can see that the component of V_2 (projected) in the direction of V_1 will be,

$$\left|V_{2_{V1}}\right| = |V_2|\cos(\theta_2 - \theta_1)$$

Next, we can manipulate this expression into the dot product form,

$$= |V_2|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$= |V_2|[(\cos\theta_1 \underset{\sim}{i} + \sin\theta_1 \underset{\sim}{j}) \bullet (\cos\theta_2 \underset{\sim}{i} + \sin\theta_2 \underset{\sim}{j})]$$

$$= |V_2|\left[\frac{V_1}{|V_1|} \bullet \frac{V_2}{|V_2|}\right] = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right] = \frac{V_1 \bullet V_2}{|V_1|} = V_2 \bullet \lambda_{V_1}$$

Or more generally,

$$\left|V_{2_{V1}}\right| = |V_2|\cos(\theta_2 - \theta_1) = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

$$\therefore |V_2|\cos(\theta_2 - \theta_1) = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

$$\therefore \cos(\theta_2 - \theta_1) = \left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

*Note that the dot product also works in 3D, and similar proofs are used.

• In Scilab,

```
A = [1 2 3];
B = [4 5 6];
dot = A'*B;
```

31.1.2 Cross Product

• First, consider an example,

$$F = (-6.43\underline{i} + 7.66\underline{j} + 0\underline{k})N$$

$$d = (2\underline{i} + 0\underline{j} + 0\underline{k})m$$

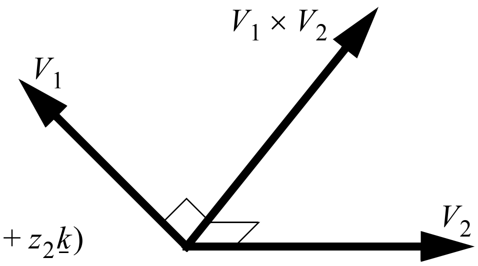
$$M = d \times F = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2m & 0m & 0m \\ -6.43N & 7.66N & 0N \end{bmatrix}$$

NOTE: note that the cross product here is for the right hand rule coordinates. If the left handed coordinate system is used F and d should be reversed.

$$\therefore M = (0m0N - 0m(7.66N))\underline{i} - (2m0N - 0m(-6.43N))\underline{j} + (2m(7.66N) - 0m(-6.43N))\underline{k} = 15.3\underline{k}(\text{mN})$$

NOTE: there are two things to note about the solution. First, it is a vector. Here there is only a z component because this vector points out of the page, and a rotation about this vector would rotate on the plane of the page. Second, this result is positive, because the positive sense is defined by the vector system. In this right handed system find the positive rotation by pointing your right hand thumb towards the positive axis (the 'k' means that the vector is about the z-axis here), and curl your fingers, that is the positive direction.

• The basic properties of the cross product are, The cross (or vector) product of two vectors will yield a new vector perpendicular to both vectors, with a magnitude that is a product of the two magnitudes.



$$V_1 \times V_2 = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$V_1 \times V_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

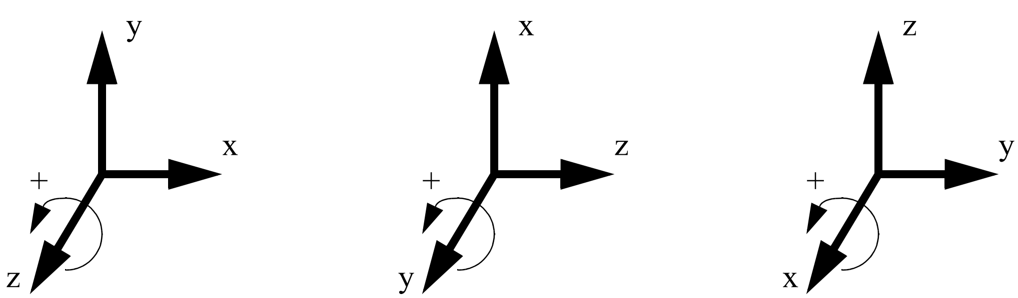
$$V_1 \times V_2 = (y_1z_2 - z_1y_2)\underline{i} + (z_1x_2 - x_1z_2)\underline{j} + (x_1y_2 - y_1x_2)\underline{k}$$

We can also find a unit vector normal 'n' to the vectors 'V1' and 'V2' using a cross product, divided by the magnitude.

$$\lambda_n = \frac{V_1 \times V_2}{|V_1 \times V_2|}$$

• When using a left/right handed coordinate system,

The positive orientation of angles and moments about an axis can be determined by pointing the thumb of the right hand along the axis of rotation. The fingers curl in the positive direction.



• The properties of the cross products are, The cross product is distributive, but not associative. This allows us to collect terms in a cross product operation, but we cannot change the order of the cross product.

$r_1 \times F + r_2 \times F = (r_1 + r_2) \times F$	DISTRIBUTIVE
$r \times F \neq F \times r$ but $r \times F = -(F \times r)$	NOT ASSOCIATIVE

• In Scilab,

```
function val=crossproduct(A, B) // No function is defined so use the following
        val = [A(2) * B(3) - A(3) * B(2) ;
        A(3) * B(1) - A(1) * B(3)
        A(1) * B(2) - A(2) * B(1)];
endfunction
```

31.2 Problems

1.

32. MATRICES

Topics:

•

Objectives:

•

32.1 Introduction

• Matrices allow simple equations that drive a large number of repetitive calculations - as a result they are found in many computer applications.

• A matrix has the form seen below,

$$\begin{array}{cc} \text{n columns} & \\ \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix} & \text{m rows} \end{array}$$

If n=m then the matrix is said to be square. Many applications require square matrices. We may also represent a matrix as a 1-by-3 for a vector.

• In Scilab,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
A = [ 1 2 3 ; 4 5 6 ; 7 8 9 ]; // ; means next row
A(2, 3) // , means next element
A(1:2, 2:3) // : means a range
A(:, 2)
A(2, :)
A(2, $) // means last row or column
A($, 2)
// other commands - please use help to explore
    these
size(A); // returns the rows, columns, etc of A
ones(A); // fills the matrix with 1s
zeros(A); // fills the matrix with 0s
eye(5, 5); // creates a 5x5 identity matrix
diag(A); // gets the diagonal of matrix A
rand()
max()
rank()
cond()
spec()
trace()
// also try
B = [ 1 2 3 ];
C = [1 ; 2 ; 3 ];
D = 1:0.1:3;
E = [ B 4 ]; // adds a column to B to make E
F = [ C ; 4 ]; // add a row to C to make F
length(C); // the rows in C
G = A(2:3 , 1:2); // extracts a 2x2 from A
H = B. * C; // element wise operation, a 3x3 results
J = B^2; // same as B*B
K = B.^2; // each element is squared
```

32.1.1 Basic Matrix Operations

• Matrix operations are available for many of the basic algebraic expressions, examples are given below. There are also many restrictions - many of these are indicated.

$$A = 2 \qquad B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \qquad C = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix} \qquad D = \begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix} \qquad E = \begin{bmatrix} 24 & 25 & 26 \end{bmatrix}$$

Addition/Subtraction

$$A + B = \begin{bmatrix} 3+2 & 4+2 & 5+2 \\ 6+2 & 7+2 & 8+2 \\ 9+2 & 10+2 & 11+2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3+12 & 4+13 & 5+14 \\ 6+15 & 7+16 & 8+17 \\ 9+18 & 10+19 & 11+20 \end{bmatrix}$$

$$B + D = \text{not valid}$$

$$B - A = \begin{bmatrix} 3-2 & 4-2 & 5-2 \\ 6-2 & 7-2 & 8-2 \\ 9-2 & 10-2 & 11-2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3-12 & 4-13 & 5-14 \\ 6-15 & 7-16 & 8-17 \\ 9-18 & 10-19 & 11-20 \end{bmatrix}$$

$$B - D = \text{not valid}$$

$$A \cdot B = \begin{bmatrix} 3(2) & 4(2) & 5(2) \\ 6(2) & 7(2) & 8(2) \\ 9(2) & 10(2) & 11(2) \end{bmatrix} \qquad \frac{B}{A} = \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\ \frac{6}{2} & \frac{7}{2} & \frac{8}{2} \\ \frac{9}{2} & \frac{10}{2} & \frac{11}{2} \end{bmatrix}$$

$$B \cdot D = \begin{bmatrix} (3 \cdot 21 + 4 \cdot 22 + 5 \cdot 23) \\ (6 \cdot 21 + 7 \cdot 22 + 8 \cdot 23) \\ (9 \cdot 21 + 10 \cdot 22 + 11 \cdot 23) \end{bmatrix} \qquad D \cdot E = 21 \cdot 24 + 22 \cdot 25 + 23 \cdot 26$$

$$B \cdot C = \begin{bmatrix} (3 \cdot 12 + 4 \cdot 15 + 5 \cdot 18) & (3 \cdot 13 + 4 \cdot 16 + 5 \cdot 19) & (3 \cdot 14 + 4 \cdot 17 + 5 \cdot 20) \\ (6 \cdot 12 + 7 \cdot 15 + 8 \cdot 18) & (6 \cdot 13 + 7 \cdot 16 + 8 \cdot 19) & (6 \cdot 14 + 7 \cdot 17 + 8 \cdot 20) \\ (9 \cdot 12 + 10 \cdot 15 + 11 \cdot 18) & (9 \cdot 13 + 10 \cdot 16 + 11 \cdot 19) & (9 \cdot 14 + 10 \cdot 17 + 11 \cdot 20) \end{bmatrix}$$

$$\frac{B}{C}, \frac{B}{D}, \frac{D}{B}, \text{etc} = \text{not allowed (see inverse)}$$

Note: To multiply matrices, the first matrix must have the same number of columns as the second matrix has rows.

32.1.2 Determinants

• Determinants give a 'magnitude product' of a matrix. This can be though of as a general magnitude of the matrix.
• To find a determinant the matrix must must be square.
•

• For a 2 by 2 matrix.

Determinant

$$|B| = 3 \cdot \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} - 4 \cdot \begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} + 5 \cdot \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = 3 \cdot (-3) - 4 \cdot (-6) + 5 \cdot (-3) = 0$$

$$\begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} = (7 \cdot 11) - (8 \cdot 10) = -3$$

$$\begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} = (6 \cdot 11) - (8 \cdot 9) = -6$$

$$\begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = (6 \cdot 10) - (7 \cdot 9) = -3$$

$$|D|, |E| = \text{not valid (matrices not square)}$$

• For a 3 by 3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

• Higher order matrices follow a similar pattern. For example a 4th order matrix has the pattern,

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

32.1.3 Transpose

$$B^T = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix} \qquad D^T = \begin{bmatrix} 21 & 22 & 23 \end{bmatrix} \qquad E^T = \begin{bmatrix} 24 \\ 25 \\ 26 \end{bmatrix}$$

32.1.4 Adjoint Matrices

$$||B|| = \begin{bmatrix} \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} & -\begin{vmatrix} 6 & 8 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 9 & 11 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 9 & 10 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \end{bmatrix}^T$$

The matrix of determinant to the left is made up by getting rid of the row and column of the element, and then finding the determinant of what is left. Note the sign changes on alternating elements.

$||D||$ = invalid (must be square)

32.1.5 Inverse Matrices

$$D = B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve this equation for x,y,z we need to move B to the left hand side. To do this we use the inverse.

$$B^{-1}D = B^{-1} \cdot B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1}D = I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1} = \frac{||B||}{|B|}$$

In this case B is singular, so the inverse is undetermined, and the matrix is indeterminate.

$$D^{-1} = \text{invalid (must be square)}$$

• Some Scilab,

```
A = [ 1 2 3 ; 4 5 6 ; 7 8 9 ];
B = [ 10 ; 11 ; 12 ];
A’ // transpose
det(A) // determinant
inv(A) // inverse
A^-1 // also inverse
spec(A)
[D, X] = bdiag(A)
linsolve(A, B)
A * B
B * A
A + B // will not work
```

32.1.6 Identity Matrix

This is a square matrix that is the matrix equivalent to '1'.

$$B \cdot I = I \cdot B = B$$

$$D \cdot I = I \cdot D = D$$

$$B^{-1} \cdot B = I$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc=I}$$

32.1.7 Eigenvalues

• The eigenvalue of a matrix is found using,

$$|A - \lambda I| = 0$$

32.1.8 Eigenvectors

32.2 Matrix Applications

32.2.1 Solving Linear Equations with Matrices

• Note: if the determinant of a matrix is 0, the matrix is singular and there is no solution for the linear equations

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

The solution is found using the inverse,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \frac{\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \frac{\begin{bmatrix} 15 - 8 \\ -10 + 8 \end{bmatrix}}{3 - 2} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

• In Scilab,

```
A = [1 1 ; 2 3];
B = [5; 8];
X = inv(A) * B
X = A \ B; // another way to solve linear equations
r = B - A * x; // The residual should be 0
```

• We can solve systems of equations using the inverse matrix,

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the matrix, then rearrange, and solve.

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

• We can solve systems of equations using Cramer's rule (with determinants),

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with B.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

32.2.2 Gauss-Jordan Row Reduction

- In many ways Gauss-Jordan is a form of substitution based upon rearranging equations into an upper-right triangular form.
- The general method works top to bottom ......

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

eliminate values in the first column by multiplying by a factor and subtracting from the first row,

$$-\frac{1}{2} \quad \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 - \frac{1}{2}2 & 1 - \frac{1}{2}3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 - \frac{1}{2}8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Finally, solve for the values starting with the last row and work towards the top row.

$$-0.5y = 1 \qquad y = -2$$

$$1x + 1(-2) = 5 \qquad x = 7$$

32.2.3 Cramer's Rule

Given,

$$x + y = 5$$

$$2x + 3y = 8$$

The equations can be written in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

To find x,

$$x = \frac{\begin{vmatrix} 5 & 1 \\ 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{15 - 8}{3 - 2} = 7$$

To find y,

$$y = \frac{\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{8 - 10}{3 - 2} = -2$$

```
x = det([5 1 ; 8 3]) / det([1 1 ; 2 3])
y = det([1 5 ; 2 8]) / det([1 1 ; 2 3])
```

32.2.4 Triple Product

When we want to do a cross product, followed by a dot product (called the mixed triple product), we can do both steps in one operation by finding the determinant of the following. An example of a problem that would use this shortcut is when a moment is found about one point on a pipe, and then the moment component twisting the pipe is found using the dot product.

$$(d \times F) \bullet u = \begin{vmatrix} u_x & u_y & u_z \\ d_x & d_y & d_z \\ F_x & F_y & F_z \end{vmatrix}$$

32.2.5 Gauss-Siedel

32.3 Problems

1. Perform the vector operations below,

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Cross Product $A \times B =$

Dot Product $A \bullet B =$

ANS.

$A \times B = (-4, 17, -10)$

$A \bullet B = 13$

2. Perform the following matrix calculations.

a) $$\begin{bmatrix} a & b & c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix}$$

ans.

not solvable

b) $$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$$

$ad - bc$

c) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$$\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

3. Perform the matrix operations below.

$$\begin{vmatrix} 2 & 0 \\ -6 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix}$$

ans.
=6

=12

=0

Multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} =$$

Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Inverse

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} =$$

ANS.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 36$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -0.833 & 0.167 & 0.167 \\ 0.167 & -0.333 & 0.167 \\ 0.5 & 0.167 & -0.167 \end{bmatrix}$$

4. Solve the following equations using matrices,

$$\begin{aligned} 5x - 2y + 4z &= -1 \\ 6x + 7y + 5z &= -2 \\ 2x - 3y + 6z &= -3 \end{aligned}$$

ANS.
x= 0.273
y= -0.072
z= -0.627

5. Solve the following system of equations using a) substitution, b) matrices.

$$x + 2y + 3z = 5$$
$$x + 4y + 8z = 0$$
$$4x + 2y + z = 1$$

ans. $x = -7$

$y = 18.75$

$z = -8.5$

6. Find the dot product, and the cross product, of the vectors A and B below.

$$A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

ans. $A \bullet B = xp + yq + zr$

$$A \times B = \begin{bmatrix} yr - qz \\ zp - xr \\ xq - py \end{bmatrix}$$

33. GRAPHING

Topics:

-

Objectives:

-

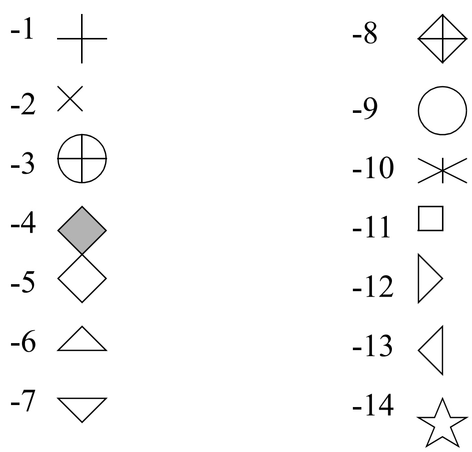
33.1 Introduction

- coordinates
- types - line, bar, scatter, x-y, chart
- curves vs. data
- SCILAB examples

• With Scilab,

```
t = [ 1 2 3 4 5 ];
f = [ 1 4 9 16 25 ];
plot2d(t, f);
// pause here and look at the graph
xbasc(); // clears the screen
plot2d(t, f, style=-1, leg='test graph')
xtitle('time (min)');
```

• Scilab graph styles,



33.2 Graphing Functions

• In Scilab,

```
deff('[y]=f(t)', [y=%e^(-5*t)']);
f(2)
t = (0:0.1:10);
fplot2d(t, f);

// OR

t = (0:0.1:10);
y=%e^(-5*t);
plot2d(t, f);
```

33.3 LOG Plots

• In Scilab,

```
f = [1, 10, 100, 1000, 10000];
G = [10, 10, 20, 100, 1000];
plot2d(log10(f), log10(G), style=-1);
// OR
plot2d(f, G, style=-1, logflag='11');
```

33.4 Multiple Plots

• In Scilab,

```
t = [0: 0.1: 6];
y = sin(t) + sin(3 * t);
z = sin(t) + cos(t);
xset('window', 0); plot(t, y);
xset('window', 1); plot(t, z);
```

33.5 Other Items of Interest

• In Scilab,

```
xbasimg(0, 'filename.eps);
```

33.6 Problems

1. Plot the following function using 20 datapoints in Scilab from -10 to 10.

$$y(t) = e^{-5t}\sin(3t)$$

2. Draw a scatter plot in Scilab for the following (x, y) data.

(1,1) (2, 3) (2, 5) (1, 7) (5, 8)

33.7 Challenge Problem

1. Draw a histogram (bar chart) for the following raw data.

1, 4, 7, 3, 5, 2, 4, 3, 7, 9, 7, 3, 4, 1, 7, 5, 2, 4, 3, 6, 9, 5, 1, 5, 2, 3, 8

34. PROGRAMMING

Topics:

-

Objectives:

-

34.1 Overview

- brief summary of topic(s)
- category: review / fundamental / application / research

34.2 Introduction

• Some useful program elements
  - 'clear' empties the screen
  - 'exec' executes a file as a script.
  - 'chdir' changes a working directory
  - 'getf('functionfile') - gets a function from a working directory
  - pwd - returns the current working directory
  - 'xbasc() - clears a graphics window

• For-loops in Scilab,

```
for i=0:0.1:6 // 0 to 6 in 0.1 steps
	// loop code goes here
end

// OR using vectors
V=[0:0.1:6];
for i=V
	// loop code goes here
end
```

• While-loops in Scilab,

```
i = 0;
while(i <= 5),
	i = i + 1;
	i
end
i
```

• If-then in Scilab,

```
if(__logical__),
	// some code
else
	// other code
end

// >, >=, <, <=, ==, ~= Basic comparisons
// &, |, ~ Basic Boolean operators
```

• Functions in Scilab,

```
function returnval = functionname(arg1, arg2)
        returnval = arg1 + arg2;
        if(returnval < 0),
                        returnval = 0;
                        return
        end
endfunction

functionname(5, 3)
```

• Note that when a function is defined in Scilab it is not actually run. However when the function is called it will be interpreted.
• Cases in Scilab,

```
select val,
        case 0,
          // code
        case 1,
          // more code
        else
          break
end
```

• Printing in Scilab,

```
mprintf("format string goes here %d %f \n", arg_int, arg_float)

// %d - integer
// %f - float
// \n - end of line
```

• Inputting values in Scilab,

```
a = input("input a value for a");
```

34.3 Examples

• A simple Scilab Program,

```
//
// test.sce
//
// A simple program to integrate an accelerating mass
//
// To run this in Scilab use 'File' then 'Exec'.
//
// by: H. Jack Aug 27, 2002
//

// Set the time length and step size
steps = 100;
t_end = 10;
delta_t = t_end / steps;

// Initial conditions for the motion
pos=[0];
vel=[0];
t=[0];
acc=9.81;

// Loop to integrate the motion
for i=1:steps,
        t = [t; i * delta_t];
        vel = [vel; vel($) + acc];
        pos = [pos; pos($) + vel($)*delta_t + 0.5*acc*delta_t^2];
end

// Dump the values to the screen
//[t vel pos]

// Graph the values
plot2d(t, [pos vel], [-2, -5], leg="position@velocity");
                                // leg - the legend titles
                                // style - draw lines with marks
                                // nax - grid lines for the graph
xtitle('Time (s)');

// Write values to a file (delete the existing file first)
unix("del c:\temp\data.txt");
write("c:\temp\data.txt", [t vel pos]);
```

• A simple numerical integration of the equations of motion (ME/PDM),

```
// first_order.sce
// A first order integration of an accelerating mass
// To run this in Scilab use 'File' then 'Exec'.
// by: H. Jack Sept., 16, 2002

// System component values
mass = 10;
force = 100;

x0 = 8;                         // initial conditions
v0 = 12;
X=[x0, v0];

// define the state matrix function
// the values returned are [x, v]
function foo=f(state,t)
        foo = [ state($, 2), force/mass]; // d/dt x = v, d/dt v = F/M
endfunction

// Set the time length and step size for the integration
steps = 100;
t_start = 1;
t_end = 100;
h = (t_end - t_start) / steps;

// Loop for integration
for i=1:steps,
        X = [X ; X($,:) + h*f(X, i*h)];
end
printf("The value at the end of first order integration is (x, v) = (%f, %f)\n", ...
        X($,1),                 ...
        X($,2));

// Explicit equation
function x=position(x0, v0, a0, t)
        x = (0.5 * a0 * t^2) + (v0 * t) + x0;
endfunction

function v=velocity(v0, a0, t)
        v = (a0 * t) + v0;
endfunction
```

• Integration continued,

```
printf("The value with integration is (x, v) = (%f, %f)\n", ...
                                  position(x0, v0, force/mass, t_end), ...
                                  velocity(v0, force/mass, t_end));


//
// The results should be
//      first order integration = (49710, 1002)
//      explicit               = (51208, 1012)
//
// The difference is 1498 for position and 10 for velocity. This is relatively small, but
// shows a clear case of the innacuracy of the numerical solutions.
//
// Note: increasing the number of steps increases the accuracy
//
```

• A simple Bode plot example Program (for electricals),

```
steps_per_dec = 6;
decades = 6;
start_freq = 0.1;

// The transfer function
function foo=G(w)
	D = %i * w;
	foo = (D + 5) / (D^2 + 100*D + 10000);
endfunction

fd = mopen("data.txt", "w");
for step = 0:(steps_per_dec * decades),
	f = start_freq * 10 ^ (step/steps_per_dec);// calculate the next frequency
	w = f * (2 * %pi);				// convert the frequency to radians
	[gain, phase] = polar(G(w));// get the result and convert to magnitude/angle
	gaindb = 20*log10(gain);// convert to dB
	phasedeg = 180 * phase / %pi;// convert to degrees
	mfprintf(fd, "%f, %f, %f \n", f, gaindb, phasedeg);
end
mclose(fd);


// To graph it directly
D=poly(0,'D');
h=syslin('c', (D + 5) / (D^2 + 100*D + 10000) );
bode(h, 0.1, 1000, 'Sample Transfer Function');
```

34.4 Summary

34.5 Problems

1. Write a program that chooses a random number between 1 and 100. A user will then have to guess the value. Each guess will be given a hint for higher/lower. At the end the game should report the number of guesses required.

34.6 Challenge Problems

35. PERMUTATIONS AND COMBINATIONS

Topics:

-

Objectives:

-

35.1 Introduction

• Permutations - a count of the variety of sequences

• Combinations - a count of the number of selections

• Probability - The chance something will happen.

35.2 Permutations

• the possible arrangements of objects given,

n = number of source objects
r = number of arranged objects

• Permutations - for exact definitions of not only possibilities, but also order.

$$P_r^n = \frac{n!}{(n-r)!}$$

e.g. How many permutations picking 2 cards off a deck (52)

$$\frac{(52)!}{(52-2)!} = \frac{52 \times 51 \times 50 \times \ldots}{50 \times 49 \times \ldots} = 52 \times 51$$

• permutations for n unique objects arranged into r spots

$$\frac{n!}{(n-r)!}$$

For example 5 objects in 3 spots

$$\frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60$$

• permutations for non-unique objects (using all)

For example consider 3 pairs of objects that will fill 6 spots

$$\frac{n!}{d_1!d_2!d_3!} = \frac{6!}{2!2!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8} = 90$$

35.3 Combinations

• Combinations - similar to before except order does not mater.

$$C_r^n = \frac{n!}{r!(n-r)!} \qquad \text{(n choose r)}$$

e.g. How many combinations of the first 2 cards can be picked off the deck

$$\frac{(52)!}{2!(52-2)!} = \frac{52 \times 51 \times 50 \times \ldots}{(2 \times 1)(50 \times 49 \times \ldots)} = \frac{52 \times 51}{2}$$

• Possible outcomes if ’r’ objects are taken from a group of ’n’.

$$nCr = \frac{nPr}{r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

For example consider 2 cars chosen from a set of 4

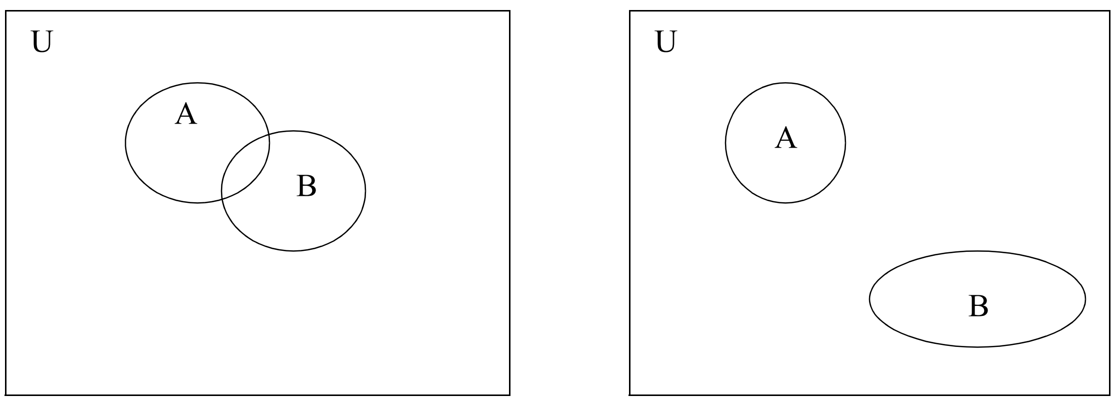
$$\frac{4 \cdot 3}{2 \cdot 1} = 6$$

35.4 Probability

• The chance ’P(A)’ some event ’A’ will happen.

• A way to figure out how chances interact

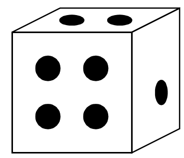
• Venn diagrams can be useful for describing interactions,

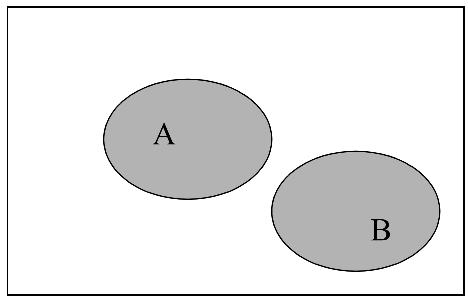


• Mutually Exclusive - Probable events can only happen as one or the other.

e.g. Only one number from 1 to 6 will come up when rolling a die.

$P(A) + P(B) + ... = 1$

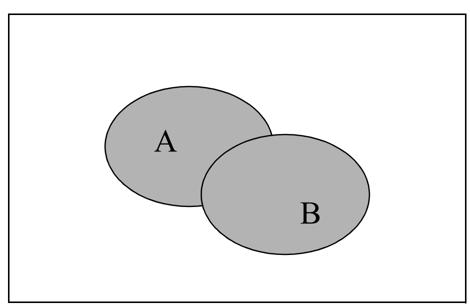




• Not Mutually Exclusive - Probable events can occur simultaneously

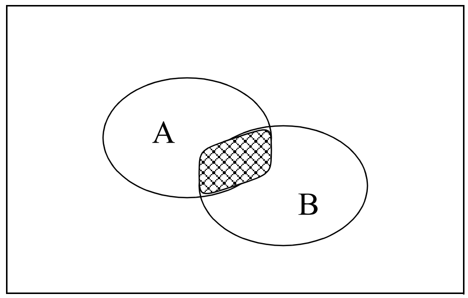
e.g. Two dice are rolled to get a number from 2 to 12, the chance that a 4 will come up is,

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 1/6 + 1/6 - 1/36$ $\quad P(A \cup B)$



• Independent Probabilities - Events will happen separately e.g. The chance of rolling a number >= 3 then <3 on a 6 sided die.

P(A and B) = P(A) * P(B) = (4/6)*(2/6) = 8/36 $P(A \cap B)$



• Dependant Probabilities - The outcome of one event effects the outcome of another event

e.g. The chance a student will pass an exam if they show up (99/100) and they know the material (7/10)

P(A and B) = P(A) * P(B when A) = (99/100)*(7/10)

• empirical probability - experimentally determine the probability with,

$$P = \frac{\text{\# of outcomes}}{\text{\# of trials}}$$

• Probability Density Function

$$P(x_1 \le x \le x_2) = \int_{x_1}^{x_2} f(x) dx$$

35.5 Problems

1. The marketing department has asked how many product permutations are possible for a new product. The product will hold three colored balls. In total there are 6 ball colors available. If balls must be used in all spots, and ball colors can be used more than once how many permutations are possible? (ans. 216)

2. A worker is packing small cereal boxes into a larger box. If the worker is picking from a set of 10 different cereal boxes, but can only place 8 in the larger box, how many combinations are possible? (ans. 45)

3. An assembly operation places two parts a board. There is a 3% chance that part A is bad, and a 2% chance that part B is bad. What is the chance the board contains a bad part? (ans. 4.94%)

4. How many ways could 5 operators be assigned to 5 workstations? (ans. 120)

5. How many ways could 8 operators be assigned to 5 workstations? (ans. 6720)

6. There are nine product, each a different color. They are to be put into 3 boxes each holding 3 products. a) How many unique package arrangements are possible? b) How many different combinations of packaged products are possible if the position does not matter? (ans. a) 362,880 b) 1680)

7. An electronics company will assemble circuit boards with interchangeable components. There are 3 places to mount the components and there are 5 types of components. Each component type may only be used once. How many different outcomes are possible? (ans. 60)

8. A toy is being manufactured to have prizes in 2 of 5 slots. How many prize layouts are possible? (ans. 10)

9. Calculate the following values.

a) $_7C_3 =$

b) $_4C_2 =$

c) $_7C_r = 21$

(ans. $= 35$

$= 6$

$r = 2$

10. There are 8 machines (A to H) waiting to be shipped. 3 of these will be tested.
a) How many combinations are possible?
b) How many of those combinations contain machine C?
c) How many combinations contain A or H, but not both?
d) If 2 trucks are loaded with 4 machines, how many distributions are possible?
e) Resolve part d) if machine A and B are in the same truck.
(ans. a) 56, b) 21, c) 30, d) 70, e) 15)

11. A carton contains 12 parts, 4 are red and 8 are green.
a) Find the probability that the first part removed is green.
b) If 3 parts are removed what is the probability that all are red?
c) Repeat b) for all green.
d) Repeat b) for one red and 2 green.
(ans. a) 2/3, b) 1/55, c) 14/55, d) 28/165)

12. Write routines to implement basic functions.

36. STATISTICS

Topics:

-

Objectives:

-

36.1 Introduction

• Descriptive vs. predictive statistics

• Why?
- Because we can't sample every part.
- Because no matter what we do, no two parts will be the same.
- Because we define a product to meet specifications and we can measure how well it conforms.
- Because differences between parts are hard (assignable causes) or impossible (chance causes) to predict.
- Because we can sample a few and draw conclusions about the whole group.

• What is the objective?
- We want to sample as little data as possible to draw the most accurate conclusions about the distribution of the values.

• Two type of statistics
- Inductive - Try to get overall variance within group. i.e. assume all of group should conform ***WE USE THIS TYPE
- Deductive - Attempt to classify differences that exist within a group (i.e. election polls).

36.2 Data Distributions

• Showing Differences
  - consider Dominos pizza that delivered < 30minutes by tolerance

Delivery times for one night

GROUPED
to nearest value

20	23	27	23	26
16	26	17	29	26
28	26	23	26	31
21	27	29	17	27

because data is grouped, 20.7 minutes becomes 21

Pizzas with black olives for 20 days

UNGROUPED
data

1	4	5	2	1
2	4	4	7	1
8	2	3	3	6
3	7	9	7	7

We can draw all with a tally sheet

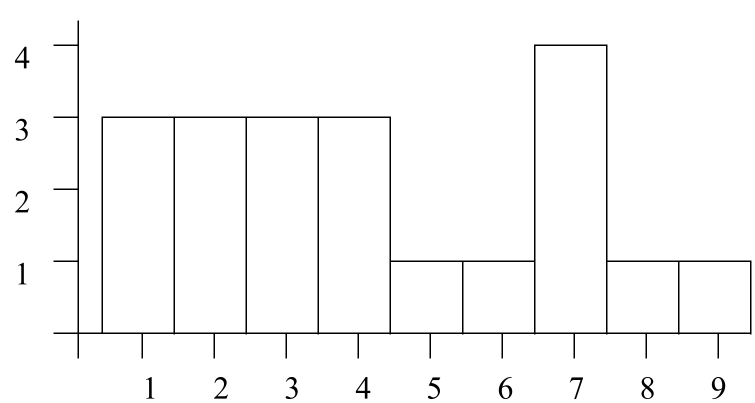
#	count	freq.
1	III	3
2	III	3
3	III	3
4	III	3
5	I	1
6	I	1
7	IIII	4
8	I	1
9	I	1

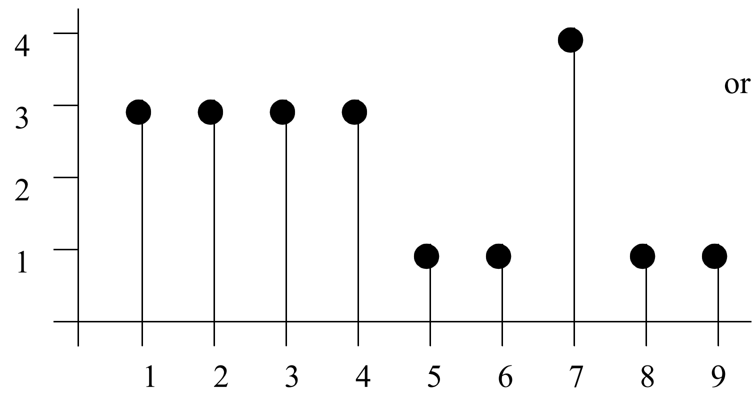
• In Scilab,

```
D = [ 3 ; 7 ; 8 ; 2 ; 5 ; 6 ; 3 ; 2 ; 9 ; 4 ; 5 ; 8 ]; // a data set
min(D); // the minimum value in D
max(D); // the maximum value in D
median(D); // the median value in D
mean(D); // the average value for the samples
find(...); // a function for looking for logical conditions
sum(D); // the sum of all elements in D
```

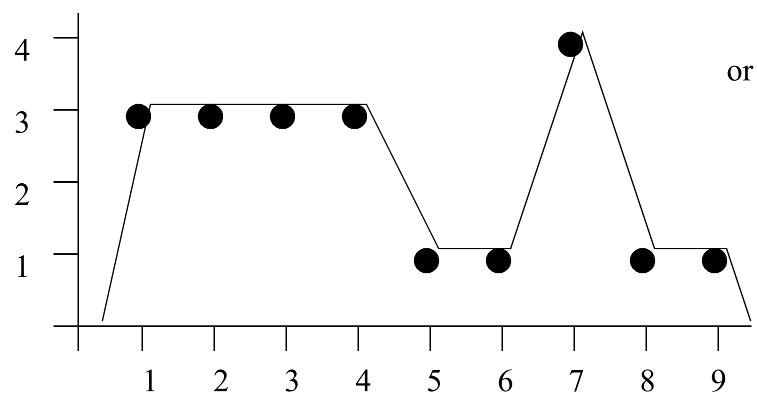
36.2.1 Histograms

• Histograms can be used to show these distributions graphically.

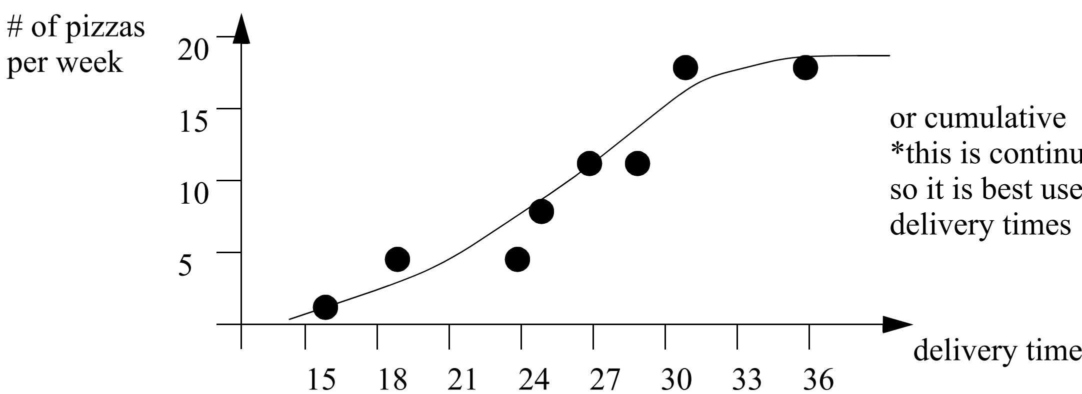




or



or polygon graph



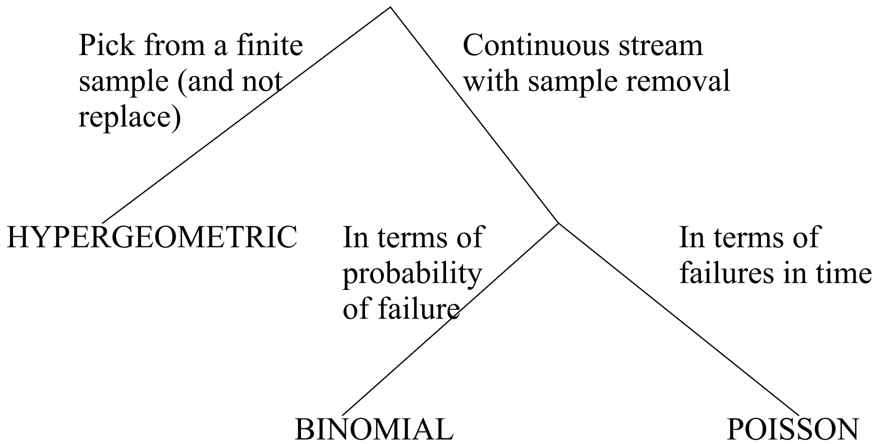
or cumulative
*this is continuous,
so it is best used for the
delivery times

- Cumulative distributions can be used to estimate the probability of an event. For example, if in the graph above we want to know how many pizzas are delivered within 25 minutes, we could read 10 (approx.) every week off the graph.

- there are typically 10 to 20 divisions in a histogram

- percentages can be used on the right axis, in place of counts.

36.3 Discrete Distributions



36.3.1 Normal Distribution

• Symbolically

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$P(x) =$$

36.3.2 Hypergeometric Distribution

Hypergeometric

$$P(d) = \frac{C_d^D C_{n-d}^{N-D}}{C_n^N}$$ (probability of d nonconforming in sample n)

basic combinatorial formula

N - number in lot
n - number in sample
D - number nonconforming in lot
d - number nonconforming in sample

36.3.3 Binomial Distribution

Binomial

$$P(d) = \frac{n!}{d!(n-d)!}(p_0)^d(q_0)^{n-d}$$ (probability of d nonconforming)

n - number in sample
d - number nonconforming in sample
p_0 - fraction nonconforming in sample
q_0 - $(1 - p_0)$

The binomial distribution is based on a simple principle of probability.

If we consider that there is a probability p_0 that something will fail, there is a probability $q_0 = 1 - p_0$ that it will pass. In statistical terms, we are looking at all possible mutually exclusive outcomes (add the probabilities) of the independent probabilities (multiply). Therefore, for a sample size of 1 (=n),

outcome #	possible outcome	probability
1	failure	p_0
2	pass	q_0

$$\therefore \sum P = p_0 + q_0 = (p_0 + q_0)^1 = p_0 + (1 - p_0) = 1$$

To take this a step farther, if there are 2 (=n) samples,

outcome #	possible outcome	probability	
1	both fail	$p_0 * p_0 = p_0^2$	
2	first fails, second passes	$p_0 * q_0$	$2(p_0 q_0)$
3	first passes, second fails	$q_0 * p_0$	
4	both pass	$q_0 * q_0 = q_0^2$	

$$\therefore \sum P = p_0^2 + 2p_0q_0 + q_0^2 = (p_0 + q_0)^2 = (p_0 + (1 - p_0))^2 = (1)^2 = 1$$

We can continue in this way, and it will eventually show a pattern emerge in the form.
This can be written as a general equation.

$$(p_0+q_0)^n = p_0^n + np_0^{n-1}q_0 + \frac{n(n-1)}{2!}p_0^{n-2}q_0^2 + \frac{n(n-1)(n-2)}{3!}p_0^{n-3}q_0^3 + \ldots + q_0^n$$

The coefficients for this method can also be found using pascals triangle. These coefficients replace the terms in front of the p_0 and q_0 terms.

n													
0							1						
1						1		1					
2					1		2 +		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1

This triangle is constructed by adding the two factors above

We can then use this chart to construct a binomial equation. Say we have a sequence of 4 events, therefor n=4

$$1 = (1)p_0^4 + (4)p_0^3q_0 + (6)p_0^2q_0^2 + (4)p_0q_0^3 + (1)q_0^4$$

Now that we have the binomial function, we can estimate the probability of various outcomes. A table is given for clarity below,

term #	meaning	probability
0	all four of the cases fail	$(1)p_0^4$
1	three of the cases fail	$(4)p_0^3q_0$
2	two of the cases fail	$(6)p_0^2q_0^2$
3	one of the cases fail	$(4)p_0q_0^3$
4	none of the cases fail	$(1)q_0^4$

36.3.4 Poisson Distribution

Poisson

$$P(c) = \frac{(np_0)^c}{c!}e^{-np_0}$$ (probability of d nonconforming)

c - count of nonconforming
np_0 - total count in sample
e = 2.718281828....

Poisson's distribution can be used to predict non-related events.

It's general form is,

$$\sum P = 1 = e^{m} \times e^{-m}$$

We can use a power factor expansion of one of the terms,

$$e^{m} = 1 + m + \frac{m^{2}}{2!} + \frac{m^{3}}{3!} + \frac{m^{4}}{4!} + \frac{m^{5}}{5!} + \ldots$$

Substitution leads to the final distribution function,

$$\therefore 1 = e^{-m}\left(1 + m + \frac{m^{2}}{2!} + \frac{m^{3}}{3!} + \frac{m^{4}}{4!} + \frac{m^{5}}{5!} + \ldots\right)$$

$$\therefore 1 = e^{-m} + me^{-m} + \frac{m^{2}}{2!}e^{-m} + \frac{m^{3}}{3!}e^{-m} + \frac{m^{4}}{4!}e^{-m} + \frac{m^{5}}{5!}e^{-m} + \ldots$$

We can now relate this back to the probability discussed before, and into a general form.

$$\therefore \sum P = 1 = P(0) + P(1) + P(2) + P(3) + \ldots$$

$$\therefore P(n) = \frac{m^{n}}{n!}e^{-m}$$

where,

P(n) = the probability of 'n' occurrences
n = the number of occurrences being considered for a probability
m = the expected number of occurrences

Next, let us consider an example of a soft drink manufacturer that inspects outgoing bottles to see if the labels are right side up. If the typical number found is 2 per hour (m=2), what is the probability that less than 4 will occur in one hour (n<4)?

$$P(\quad < 4) = P(0) + P(1) + P(2) + P(3)$$

$$\therefore P(\quad < 4) = e^{-2} + 2e^{-2} + \frac{2^2}{2!}e^{-2} + \frac{2^3}{3!}e^{-2} =$$

36.4 Other Distributions

36.4.1 Polynomial Expansions

• Binomial expansion for polynomials,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + x^n$$

36.4.2 Discrete and Continuous Probability Distributions

• The Binomial distribution is,

$$P(m) = \sum_{t \le m} \binom{n}{t} p^t q^{n-t} \qquad q = 1-p \qquad q, p \in [0, 1]$$

• The Poisson distribution is,

$$P(m) = \sum_{t \le m} \frac{\lambda^t e^{-\lambda}}{t!} \qquad \lambda > 0$$

• The Hypergeometric distribution is,

$$P(m) = \sum_{t \le m} \frac{\binom{r}{t}\binom{s}{n-t}}{\binom{r+s}{n}}$$

• The Normal distribution is,

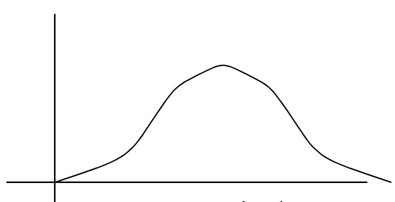
$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2}\, dt$$

36.5 Continuous Distributions

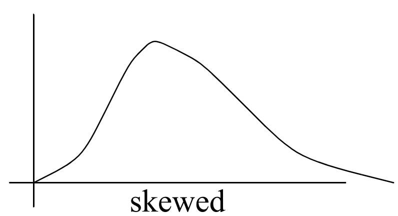
• Histograms are useful for grouped data, but in the cases where the data is continuous, we use distributions of probability.

• In general
- the area under the graph = 1.00000.......
- the graphs often stretch (asymptotically) to infinity

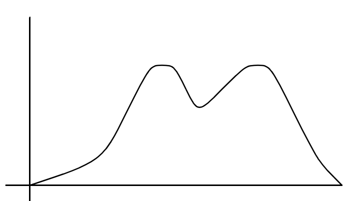
• In specific, some of the distribution properties are,



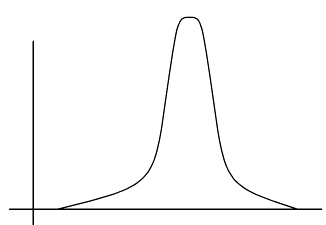
symmetrical



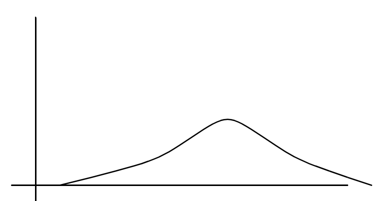
skewed



bimodal



leptokurtic



platykurtic

• In addition the centre of the distribution can vary (i.e. the average or mean)

• More on distribution later

36.5.1 Describing Distribution Centers With Numbers

• The best known method is the average. This gives the centre of a distribution

for ungrouped numbers $$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

where

$\bar{X}$ = average
X_i = a data point
n = number of points

for grouped numbers $$\bar{X} = \sum_{i=1}^{h} \frac{f_i X_i}{n}$$

where

$\bar{X}$ = average
f_i = the number of matches in range i
X_i = the central value for the range i
h = the number of ranges
n = the number of matches for all ranges

e.g. ungrouped

1, 2, 3, 4

$$\bar{X} = \frac{1+2+3+4}{4} = 2.5$$

e.g. grouped

central value	freq.
1	2
3	1
5	5
	8

$$\bar{X} = \frac{2(1) + 1(3) + 5(5)}{8} = \frac{30}{8} = 3.75$$

• Another good measure is the median

- If odd number of samples it is the middle number
- if an even number of samples, it is the average of the left and right bounding numbers

e.g. odd

1 median
2
3

even

1 median
2 $\frac{2+4}{2} = 3$
4
7

• If the numbers are grouped the median becomes

$$M_d = L_m + \left(\frac{\frac{n}{2} - cf_m}{f_m} \right) i$$

M_d = median
L_m = lower bound of the range of the median
n = number of samples overall
cf_m = cumulative frequency of all cells below L_m
f_m = frequency of median cell
i = cell interval

• Mode can be useful for identifying repeated patterns
- a mode is a repeated value that occurs the most. Multiple modes are possible.

e.g. 1 3 2 9 3 7 6 3 4 4 5
mode = 3

36.5.2 Dispersion As A Measure of Distribution

• The range of values covered by a distribution are important

- Range is the difference between the highest and lowest numbers.

e.g. 7 8 3 1 5 6 2

range = 8 - 1 = 7

• Standard deviation is a classical measure of grouping (for normal distributions??????????)

- the equation is,

$$s = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}}$$

s = standard deviation
X_i = observed value
$\bar{X}$ = average
n = number of values

e.g. 1 2 7 6 5 0 7

$$\bar{X} = \frac{1+2+7+6+5+0+7}{7} = \frac{27}{7} = 4$$

$$s = \sqrt{\frac{(1-4)^2 + (2-4)^2 + (7-4)^2 + (6-4)^2 + (5-4)^2 + (0-4)^2 + (7-4)^2}{7-1}}$$

$$= \sqrt{8.666} = 2.8$$

• When we use a standard deviation, we can estimate the distribution of the samples.

e.g

$\bar{X} = 4$
$s = 2.8$
therefore, the range 4-2.8 to 4+2.8 (1.2 to 6.8) will contain 68.26% of the samples

• By adding standard deviations to increase the range size, the percentage of samples included are,

$$\left.\begin{array}{c}\bar{X} \pm s \rightarrow 68.26\% \\ \bar{X} \pm 2s \rightarrow 95.46\% \\ \bar{X} \pm 3s \rightarrow 99.73\% \\ \text{etc....}\end{array}\right\rangle \text{Assuming a normal distribution}$$

• Other formulas for standard deviation are,

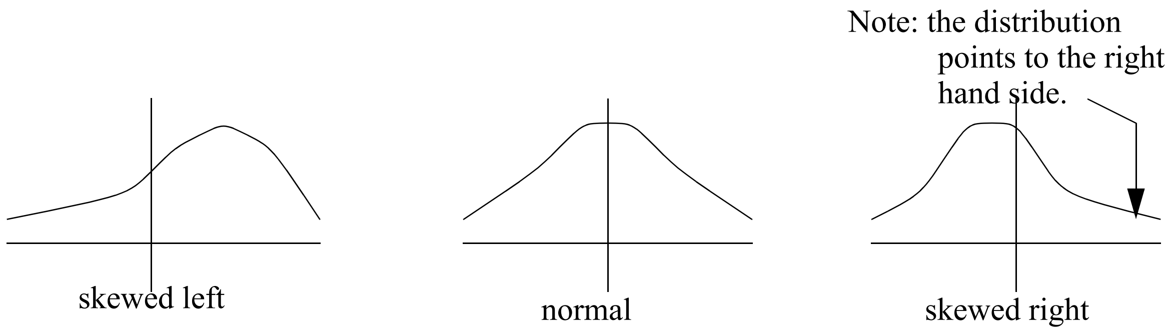
$$s = \sqrt{\frac{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}}$$

$$s = \sqrt{\frac{n\sum_{i=1}^{h} (f_i X_i^2) - \left(\sum_{i=1}^{h} f_i X_i\right)^2}{n(n-1)}}$$

h = number of cells
f_i = frequency of values in a cell

36.5.3 The Shape of the Distribution

• Skewed functions



• this lack of symmetry tends to indicate a bias in the data (and hence in the real world)

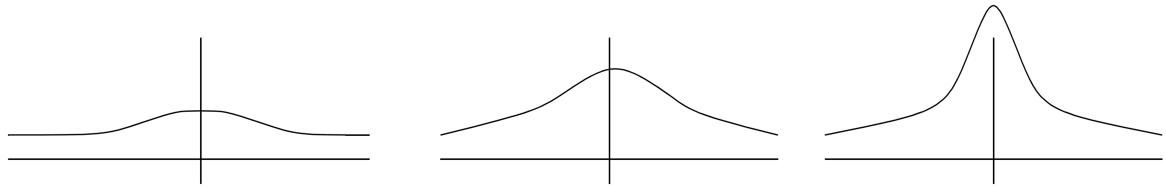
• a skew factor can be calculated

$$a_3 = \frac{\sum_{i=1}^{h} \frac{f_i (X_i - \bar{X})^3}{n}}{s^3}$$

if $a_3 = 0$ then the distribution is symmetrical (normal)
$a_3 > 0$ then skewed to the right
$a_3 < 0$ then skewed to the left
a_3 is not 0 indicates the distribution is not normal.

36.5.4 Kurtosis

• This is a peaking in the data



Platykurtic Mesokurtic (normal) Leptokurtic

$$a_4 = \frac{\sum_{i=1}^{h} \frac{f_i(X_i - \bar{X})^4}{n}}{s^4}$$

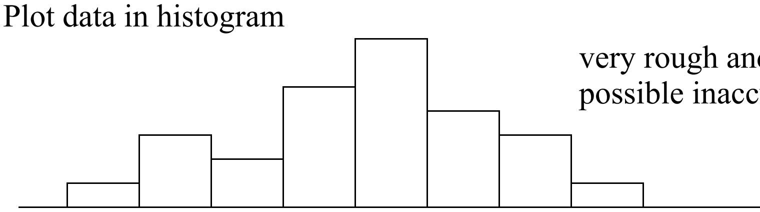
$a_4 = 3$ mesokurtic
$a_4 > 3$ leptokurtic
$a_4 < 3$ platykurtic

• This is best used for comparison to other values. i.e. you can watch the trends in the values of a_4.

36.5.5 Generalizing From a Few to Many

sample

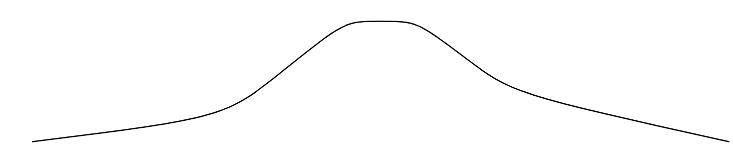
1. Randomly collect some data
2. Plot data in histogram



very rough and full of possible inaccuracies

- number of samples increases
- uncertainty decreases

3. Find average $\bar{X}$, and standard sample deviation s
4. Repeat steps 1, 2 & 3 until no longer interested.
5. Put together data from earlier samples to get much better distributions



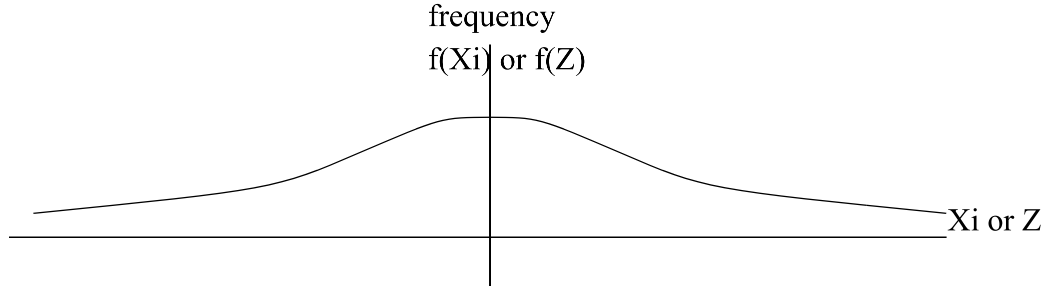
population

6. Calculate population mean μ and standard deviation σ

36.5.6 The Normal Curve

• this is a good curve that tends to represent distributions of things in nature (also called Gaussian)

• This distribution can be fitted for populations (μ, σ), or for samples ($\overline{X}$, s)

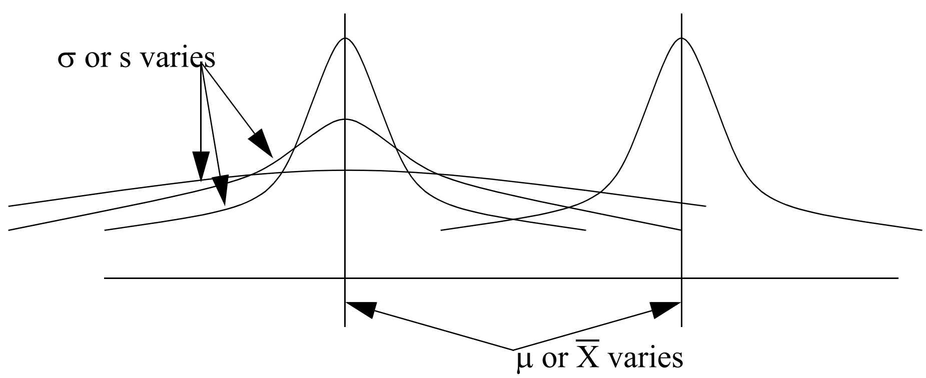


$$Z = \frac{X_i - \mu}{\sigma}$$

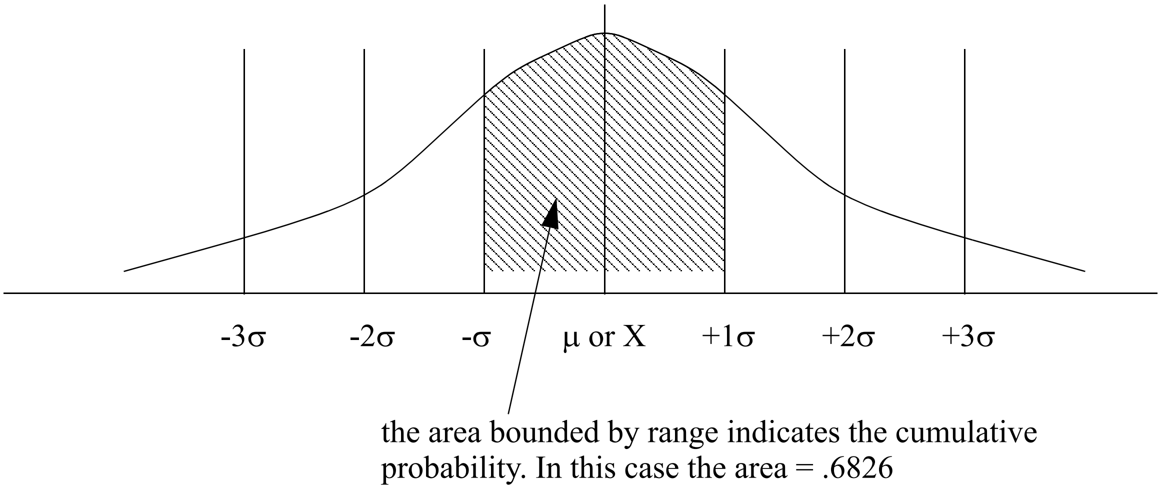
$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-Z^2}{2}}$$

• The area under the curve is 1, and therefore will enclose 100% of the population.

• the parameters vary the shape of the distribution



• The area under the curve indicates the cumulative probability of some event



μ+/- 1σ = 68.26% of the values bounded
μ+/- 2σ = 95.46% of the values bounded

between μ+1σ and μ-2σ $\frac{68.26}{2} + \frac{95.46}{2} = 81.86$ of the values are found.

*Table A in text gives integrated area under the curve so that probability may be found.

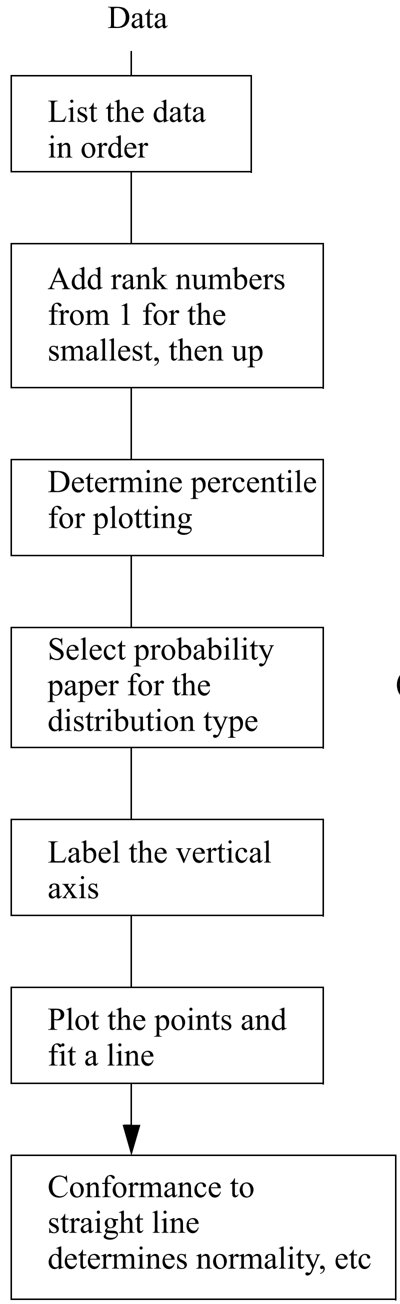
******* ADD IN MORE TO FIND AREA UNDER DISTRIBUTION

• When applied to quality $\pm 3\sigma$ are used to define a typical “process variability” for the product. This is also known as the upper and lower natural limits (UNL & LNL)

****************** LOOK INTO USE OF SYMBOLS, and UNL, LNL, UCL, LCL, etc.

36.5.7 Probability plots

• Procedure

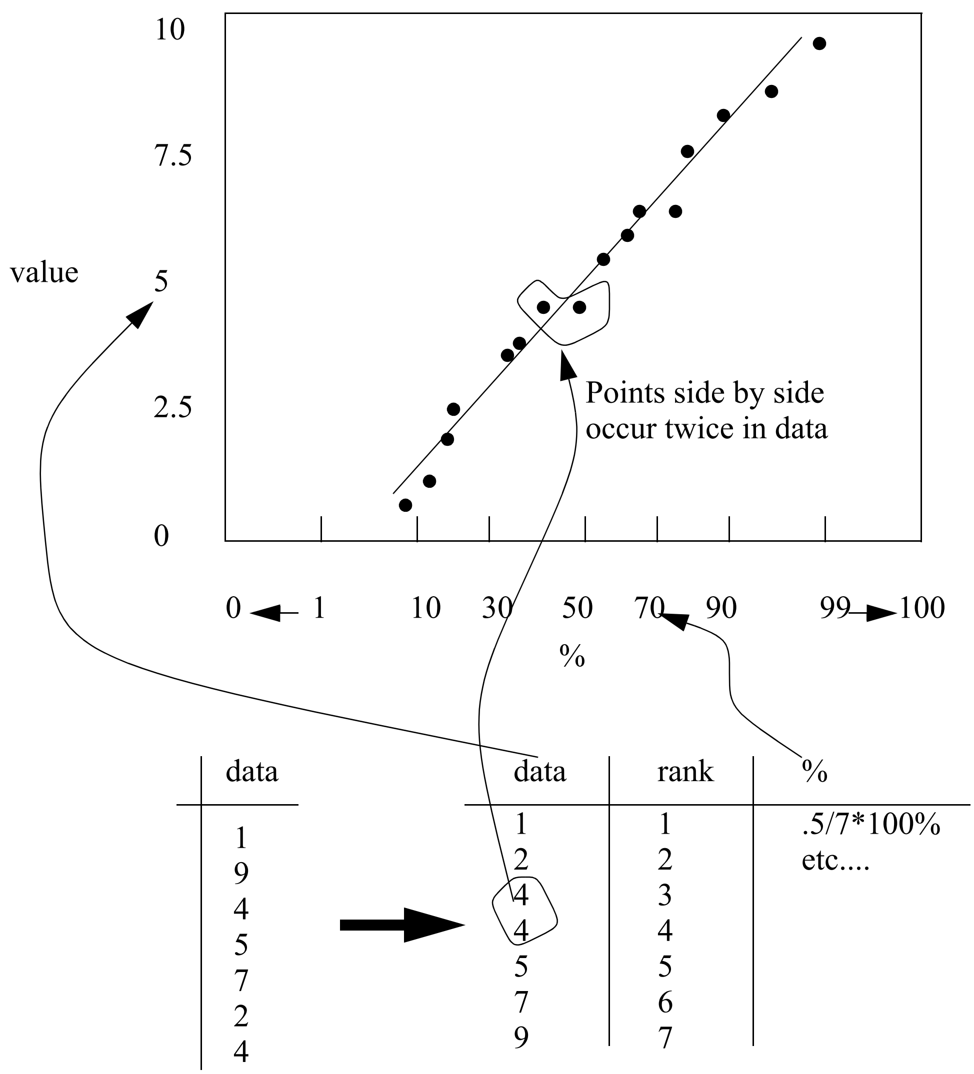


rank

$$pp = \frac{100(i - 0.5)}{n}\%$$

total # samples

(Normal in this case)



36.6 Problems

1. Write a Scilab program to compute the mean and standard deviation for the data below. It should then check to see if the values 1.0, 2.0, 3.0 are within +/- 3 standard deviations.

x
1.4
2.5
1.1
0.6
1.9
1.0
1.5
0.9
1.6
1.1
1.7
2.0
1.2
1.4
1.8

36.7 Challenege Problems

1. Write a function that generates random numbers that follow a gaussian distribution for an arbitrary mean and standard deviation. Verify the routine.
2. Use the subroutine to simulate the system described below. A production process consists of two machines. The first machine takes 3 minutes for an operation with a standard deviation of 10 seconds. The second machine takes 4 minutes with a standard deviation of 15 seconds. If the first machine finishes before the second machine, it will have to wait for the second machine to finish before the job moves on. Use Monte Carlo simulation to estimate the average cycle time.

37. RELIABILITY

Topics:
- Reliability of series and parallel system components
- MTBF, MTTF, MTTR, system availability
- FMEA

Objectives:
- To be able to estimate the reliability of a multicomponent system.

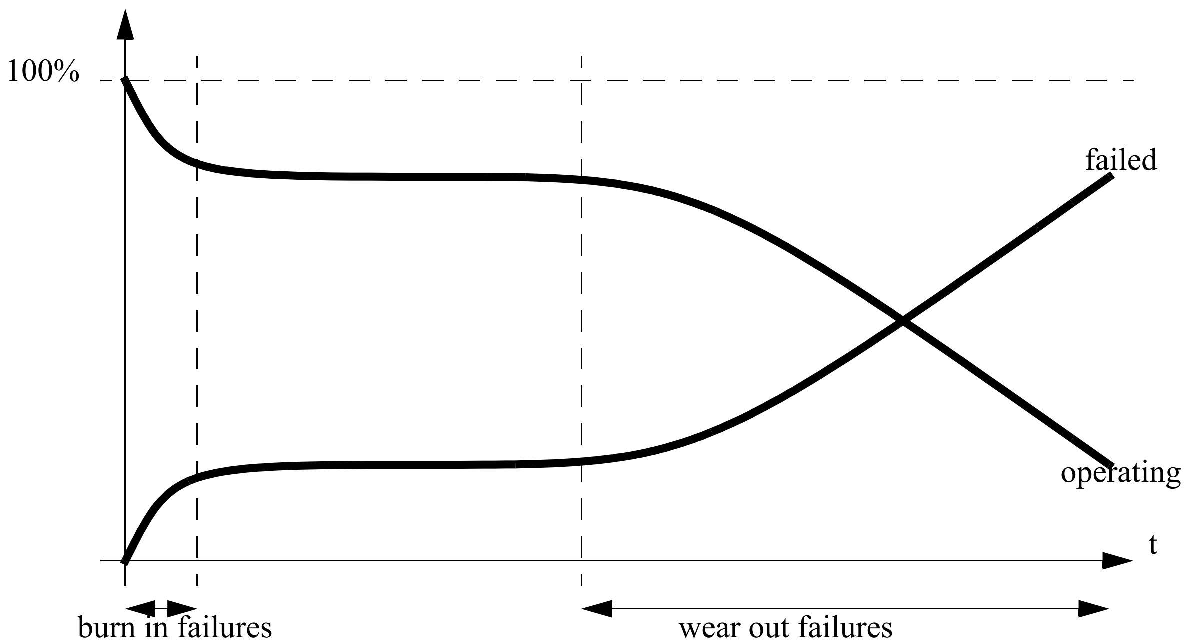
37.1 Introduction

• This chapter will assume that a system or component is functional, or not. In practice systems fail slowly, but we will consider failure to be when they cease being functional.

• No system is perfect and will fail eventually, being able to predict this allows us to determine the useable life.

• Dependability is a combination of,
  - reliability - the probability that a system operates through a given operation specification.
  - availability - the probability that the system will be available at any instant required.

• Typical failures follow the following curve.



37.2 Component Failure Rates

• Failure rate is the expected number of failures per unit time, and is shown with the constant (lambda), with the units of failures per hour.

• Basically,

$$R(t) = \frac{N(t)}{N(0)}$$

where,

$N(t)$ = the number operating at time t

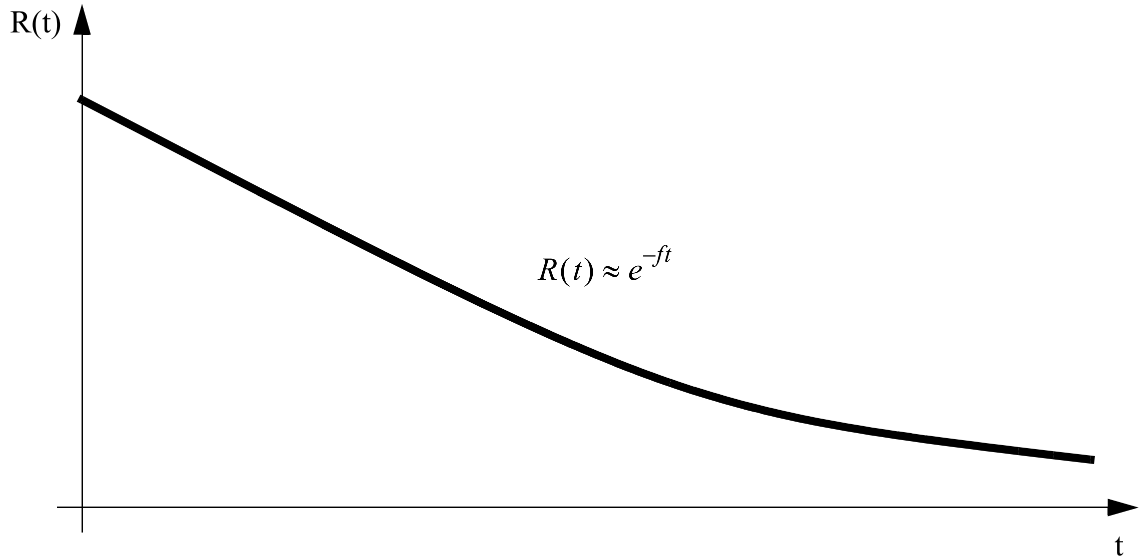
$R(t)$ = the reliability, the portion surviving over the time $[t_0, t]$

$$Q(t) = 1 - R(t)$$

where,

$Q(t)$ = unreliability

• Failures tend (but do not have to) follow exponential failure rate curves. This also suggests a failure function.



Assume a function, $R(t) = e^{-ft}$

$$\frac{d}{dt}R(t) = -fe^{-ft}$$

$$\therefore \frac{d}{dt}\frac{N(t)}{N(0)} = -fR(t)$$

$$\therefore \frac{d}{dt}\frac{N(t)}{N(0)} = -f\frac{N(t)}{N(0)}$$

$$\therefore \frac{d}{dt}N(t) = -fN(t)$$

$$\therefore f(t) = \frac{-N(t)}{\frac{d}{dt}N(t)}$$

where,

$f(t)$ = the failure rate

• A constant failure rate is the most commonly assumed. When this is the the case the failure rate is also the Mean Time Before Failure (MTBF). Note that the reliability is also the probability that the system will be operational.

$$f(t) = \frac{1}{\lambda} = \frac{1}{MTBF}$$

$$R(t) = e^{-\frac{t}{MTBF}}$$

where,

$MTBF$ = Mean Time Before Failure

- Even when the e-to-the-t function is a good model for system failure a system MTBF can be varied during system life by variations in product usage. Example of these include,
  - high heat levels
  - large loads
  - excessive stresses
  - "pot holes"
  - etc.

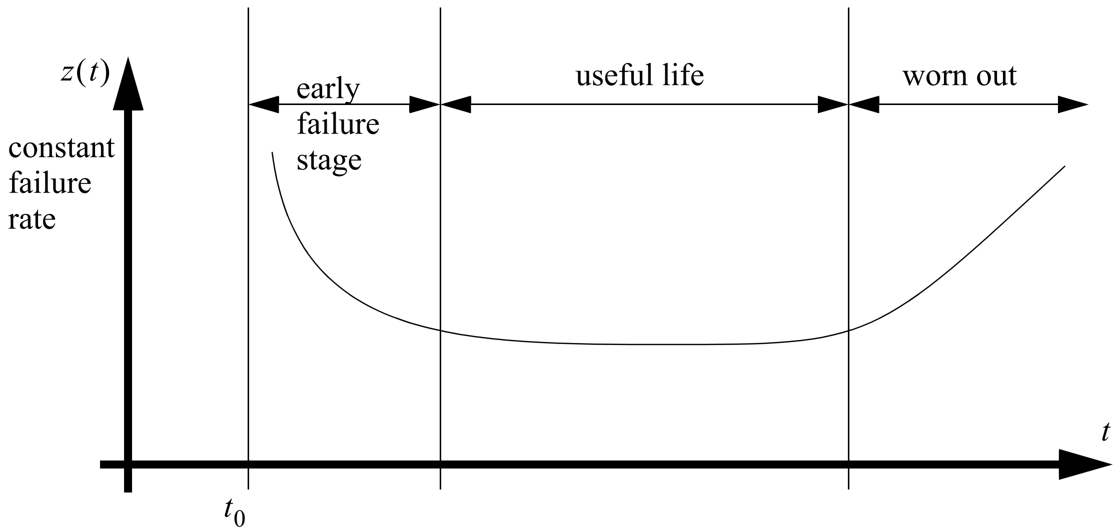
$$f(t) = -\frac{\frac{d}{dt}R(t)}{R(t)} = \frac{\frac{d}{dt}Q(t)}{1-Q(t)}$$

$$z(t) = -\frac{\frac{d}{dt}R(t)}{R(t)} = \frac{\frac{d}{dt}Q(t)}{1-Q(t)}$$

where,

$z(t)$ = the failure rate

- The bathtub curve shows typical values for the failure rate.



• The basic reliability equation can be rearranged, eventually leading to a compact expression,

$$z(t) = -\frac{\frac{d}{dt}R(t)}{R(t)}$$

$$\therefore \frac{d}{dt}R(t) = -z(t)R(t)$$

$$\therefore R(t) = e^{-\int z(t)dt}$$

During the useful life of the product, we can approximate the failure rate as linear, as reflected by the relation below,

$$\int z(t)dt \approx \lambda t$$

$$\therefore R(t) = e^{-\lambda t} = \text{The Exponential Failure Law}$$

• MTTF (Mean Time To Failure) - this is the expected time before a failure.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

where,

X = a random variable

$E[X]$ = expected value

$f(x)$ = a probability density function

$$MTTF = \int_0^{\infty} tf(t)dt$$

Given the probability density function, and using integration by parts we can find the relationship between the MTTF, and reliability.

$$f(t) = \frac{d}{dt}Q(t)$$

$$\therefore MTTF = \int_0^{\infty} t\frac{d}{dt}Q(t)dt = -\int_0^{\infty} t\frac{d}{dt}R(t)dt = [-tR(t) + \int R(t)dt]\Big|_0^{\infty} = \int_0^{\infty} R(t)dt$$

• The MTTR (Mean Time To Repair) for a system is the average time to repair a system. This is not simple to determine and often is based on experimental estimates.

$$MTTR = \frac{1}{\mu}$$

where,

μ = the repair rate $= \dfrac{\text{number of repairs}}{\text{time period for all repairs}}$

• The MTTF and MTTR both measure the time that the system is running between repairs, and the time the system is down for repairs. But, they must be combined for the more useful measure MTBF (Mean Time Before Failure),

$$MTBF = MTTF + MTTR$$

• The difference between MTBF and MTTR is often small, but when critical the difference must be observed.

• availability is the chance that at any time a system will be operational. This can be determined experimentally, or estimated. For a system that is into it's useful lifetime, this can be a good measure. Note that at the beginning, and end of its life, this value will be changing, and will not be reliable.

$$A(t) = \frac{t_o}{t_o + t_r} = \frac{MTTF}{MTTF + MTTR} = \frac{MTTF}{MTBF} = \frac{1}{1 + \frac{\lambda}{\mu}}$$

where,

$A(t) =$ probability that a system will be available at any time

$t_o =$ hours in operation over a time period

$t_r =$ hours in repair over a time period

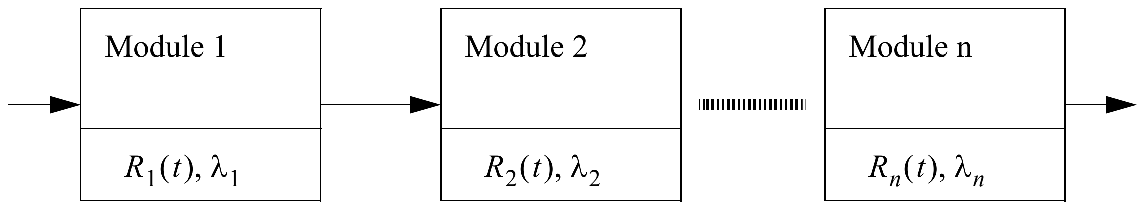
37.3 Serial System Reliabilty

• Fault Coverage is the probability that a system will recover from a failure. This can be derived approximately by examining the design, and making reliable estimates. This number will be difficult to determine exactly because it is based on real, and often unpredictable phenomenon.

• Reliability can be determined with individual system components as a function of probabilities. The two main categories of systems are series, and parallel (redundant). In the best case a high reliability system would have many parallel systems in series.

• In terms of design, a system designer must have an intuitive understanding of the concept of series/parallel functions.

• We can consider a series system where if any of the units fails, then the system becomes inoperative. Here the reliabilities of each of the system components is chained (ANDed) together.



$$R_s(t) = (R_1(t))(R_2(t))\ldots(R_n(t)) = \prod_{i=1}^{n} R_i(t)$$

where,

$R_s(t)$ = the reliability of a series system at time t

$R_i(t)$ = the reliability of a unit at time t

Now, consider the exponential failure law presented before. If each element in a system observes this law, then we can get an exact value of reliability.

$$R_s(t) = (e^{-\lambda_1 t})(e^{-\lambda_2 t})\ldots(e^{-\lambda_n t}) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n} \lambda_i t}$$

Note: this form is very nice.

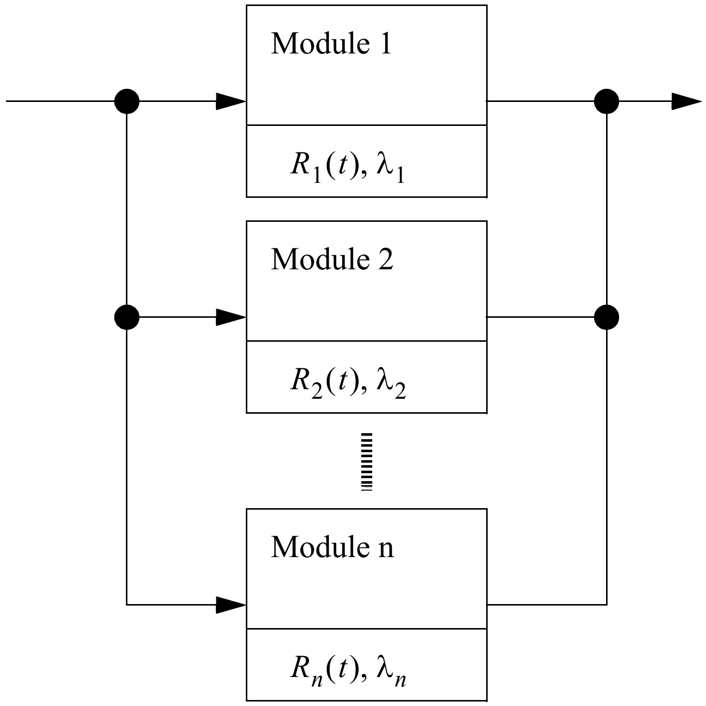
37.4 Parallel System Reliability

• When a ’parallel’ component fails the reliability of the overall system is reduced, but the system remains completely or partially functional.

• This type of reliability adds cost, so it is normally only used in critical systems where failure is not acceptable.

• Examples of systems using parallel reliability include,
- brakes on a car - 4 brakes
- electronic brakes, also have mechanical backups
- lights - in dark places multiple bulbs are used so a failed bulb does not leave it dark.

• If any of the units fails the system will continue to operate. Failure will only come when all of the modules fail. Here we are concerned with complements of the chained unreliabilities.



$$Q_p(t) = (Q_1(t))(Q_2(t))\ldots(Q_n(t)) = \prod_{i=1}^{n} Q_i(t)$$

$$R_p(t) = 1 - Q_p(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

Note: The parallel form will not result in a simple closed form as it did with the series case.

where,

$Q_s(t)$ = the unreliability of a parallel system at time t

$Q_i(t)$ = the unreliability of a module at time t

$R_p(t)$ = the reliability of a parallel system at time t

$R_i(t)$ = the unreliability of a module at time t

- also consider the case of a parallel system that requires ‘m’ of ‘n’ identical modules to be functional, such as a hybrid system, or a voting system that needs two out of three functional units. The student will consider the binomial form of the probabilities.

$$R_{m;n}(t) = \sum_{i=0}^{n-m} \binom{n}{i} (R(t))^{(n-i)} (1 - R(t))^{i}$$

where,

$R_{m;n}(t)$ = reliability of a system that contains m of n parallel modules

$R(t)$ = the reliability of the modules at time t

$\binom{n}{i} = \frac{n!}{(n-i)!i!}$ = the binomial operator (we can also use Pascal‘s triangle)

• keep in mind that many systems are a combination of series and parallel units, to find the total reliability, calculate the reliability of the parallel units first, and then calculate the series reliability, replacing the parallel units with their grouped reliability.

37.5 Formal Analysis Techniques

37.5.1 Failure Modes and Effects Analysis (FMEA)

• Estimates overall reliability of a detailed or existing product design in terms of probability of failure

• basically, each component is examined for failure modes, and the effects of each failure is considered. In turn, the effects of these failures on other parts of the system is considered.

• the following is a reasonable FMEA chart.

Critical Components	Failure Probability	Failure Mode	Number of Failures by Mode	EFFECTS	
				Critical	Non critical
car brakes (car in motion)	10^-4	disengage	10	1x10^-5	
		engage	5	5x10^-6	
		weaken	85		X
car brakes (car parked)	10^-6	disengage	40	4x10^-7	
		engage	30		X
		weaken	30		X

• the basic steps to filling one out is,

1. consider all critical components in a system. These are listed in the critical items column.
2. If a component has more than one operation mode, each of these should be considered individually.
3. estimate failure probability based on sources such as those listed below. Error bounds may also be included in the FMEA figures when numbers are unsure. These figures are entered in the “Failure Probability” column.
   - historical data for similar components in similar conditions
   - published values
   - experienced estimates
   - testing
   - etc.
4. The failures in a particular operation mode can take a number of forms. Therefore, each mode of failure for a system is considered and its % of total failures is broken down.
5. In this case the table shows failures divided into critical/non-critical (others are possible). The effects are considered, and in the event of critical failures the probabilities are listed and combined to get the overall system reliability.

• Suitable applications include,

- analyze single units or failures to target reliability problems.
- identify,
  - redundant and fail-safe design requirements

- single item failure modes
- inspection and maintenance requirements
- components for redesign

• This technique is very complete, but also time consuming.

• not suited to complex systems where cascaded errors may occur.

37.6 References and Bibliography

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37.7 Problems

1. How are series and parallel reliability different?
ans. Series reliability means that a failure of any unit will cause the entire group to fail. Parallel reliability means that there is some redundancy.

2. A set of 4 production machines are running in parallel. The first two machines have a MTBF of 100 hours and a MTTR of 3 hours. The second two machines have a MTBF of 150 hours and a MTTR of 15 hours. What is the total MTTF for the system?

3. Write a program that will accept test data to determine the Exponential Failure Equation coefficient. This should then be used to calculate the MTTF.

4. Write a program that will accept MTTF for multiple components in series, or parallel, and then calculate the combined MTTF for the system.

38. CALCULUS

Topics:

•

Objectives:

•

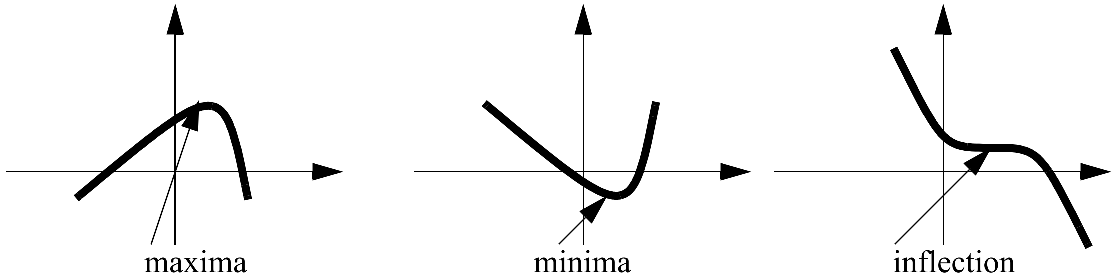
38.1 Introduction

38.2 Derivatives

• The basic definition of a derivative is,

$$\frac{d}{dt}f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

• First derivatives are often used to get the slope of a function. When this is zero the function may be at a minimum/maximum.



• Notations,

$$\frac{d}{dx}y = y'$$

$$\frac{d}{dt}y = \dot{y}$$

• The basic principles of differentiation are, Both u, v and w are functions of x, but this is not shown for brevity.

Also note that C is used as a constant, and all angles are in radians.

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(Cu) = (C)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u + v + \ldots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \ldots$$

$$\frac{d}{dx}(u^n) = (nu^{n-1})\frac{d}{dx}(u)$$

$$\frac{d}{dx}(uv) = (u)\frac{d}{dx}(v) + (v)\frac{d}{dx}(u)$$ product rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v}{v^2}\right)\frac{d}{dx}(u) - \left(\frac{u}{v^2}\right)\frac{d}{dx}(v) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\frac{d}{dx}(uvw) = (uv)\frac{d}{dx}(w) + (uw)\frac{d}{dx}(v) + (vw)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y)\frac{d}{dx}(u)$$ chain rule

$$\frac{d}{dx}(u) = \frac{1}{\frac{d}{du}(x)}$$

$$\frac{d}{dx}(y) = \frac{\frac{d}{du}(y)}{\frac{d}{du}(x)}$$

• Examples,

$$\frac{d}{dt}(t^2+t+3) = 2t+1$$

$$\frac{d}{dt}\left(\frac{t^2+t+3}{t+2}\right) = \frac{2t+1}{t+2} - \frac{t^2+t+3}{(t+2)^2}$$

• Differentiation rules specific to basic trigonometry and logarithm functions

$$\frac{d}{dx}(\sin u) = (\cos u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cos u) = (-\sin u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tan u) = \left(\frac{1}{\cos u}\right)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(e^u) = (e^u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\cot u) = (-\csc u)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sec u) = (\tan u \sec u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\csc u) = (-\csc u \cot u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tanh u) = (\mathrm{sech}\, u)^2\frac{d}{dx}(u)$$

• L'Hospital's rule can be used when evaluating limits that go to infinity.

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{\left(\frac{d}{dt}\right)f(x)}{\left(\frac{d}{dt}\right)g(x)}\right) = \lim_{x\to a}\left(\frac{\left(\frac{d}{dt}\right)^2 f(x)}{\left(\frac{d}{dt}\right)^2 g(x)}\right) = \ldots$$

• Some techniques used for finding derivatives are, Leibnitz's Rule, (notice the form is similar to the binomial equation) can be used for finding the derivatives of multiplied functions.

$$\left(\frac{d}{dx}\right)^n(uv) = \left(\frac{d}{dx}\right)^0(u)\left(\frac{d}{dx}\right)^n(v) + \binom{n}{1}\left(\frac{d}{dx}\right)^1(u)\left(\frac{d}{dx}\right)^{n-1}(v)$$

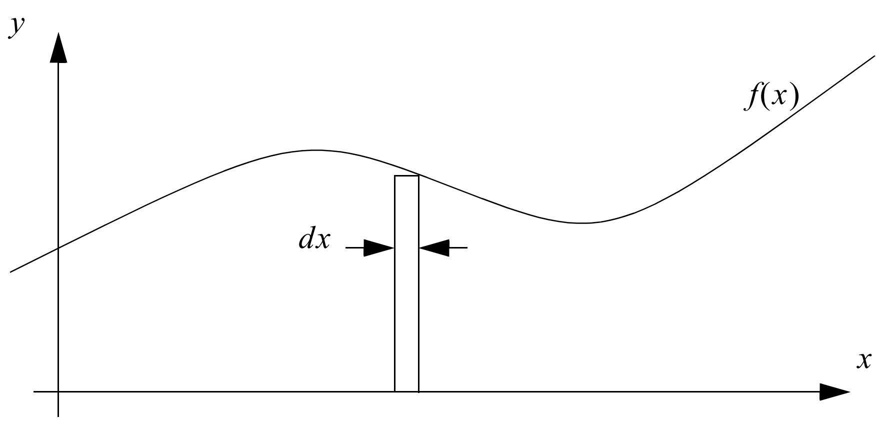
$$\binom{n}{2}\left(\frac{d}{dx}\right)^2(u)\left(\frac{d}{dx}\right)^{n-2}(v) + \ldots + \binom{n}{n}\left(\frac{d}{dx}\right)^n(u)\left(\frac{d}{dx}\right)^0(v)$$

38.3 Integrals

• Integrals are often referred to as anti-derivatives

• definite integrals have boundaries defines. Indefinitie integrals do not have boundaries defines and a constant must be added to the result.

• To set up integrals use integration elements (aka slices),



$$dA = width \cdot height = dx \cdot f(x)$$
$$A = \int f(x)dx$$

• Some basic properties of indefinite integrals (no given start and end limits) include, In the following expressions, u, v, and w are functions of x. in addition to this, C is a constant. and all angles are radians.

$$\int C dx = ax + C$$

$$\int Cf(x) dx = C\int f(x) dx$$

$$\int (u + v + w + \dots) dx = \int u dx + \int v dx + \int w dx + \dots$$

$$\int u dv = uv - \int v du = \text{ integration by parts}$$

$$\int f(Cx) dx = \frac{1}{C}\int f(u) du \qquad u = Cx$$

$$\int F(f(x)) dx = \int F(u)\frac{d}{du}(x) du = \int \frac{F(u)}{f'(x)} du \qquad u = f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int e^x dx = e^x + C$$

• Some of the trigonometric integrals are,

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int (\sin x)^2 dx = -\frac{\sin x \cos x + x}{2} + C$$

$$\int (\cos x)^2 dx = \frac{\sin x \cos x + x}{2} + C$$

$$\int (\sin x)^3 dx = -\frac{\cos x((\sin x)^2 + 2)}{3} + C$$

$$\int (\cos x)^3 dx = \frac{\sin x((\cos x)^2 + 2)}{3} + C$$

$$\int (\cos x)^4 dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\int \cos x(\sin x)^n dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \tanh x dx = \ln(\cosh x) + C$$

$$\int x \cos(ax) dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax) + C$$

$$\int x^2 \cos(ax) dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

• Some other integrals of use that are basically functions of x are,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int (a+bx)^{-1} dx = \frac{\ln(a+bx)}{b} + C$$

$$\int (a+bx^2)^{-1} dx = \frac{1}{2\sqrt{(-b)a}} \ln\left(\frac{\sqrt{a}+2\sqrt{-b}}{\sqrt{a}-x\sqrt{-b}}\right) + C,\ a>0,\ b<0$$

$$\int x(a+bx^2)^{-1} dx = \frac{\ln(bx^2+a)}{2b} + C$$

$$\int x^2(a+bx^2)^{-1} dx = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \operatorname{atan}\left(\frac{x\sqrt{ab}}{a}\right) + C$$

$$\int (a^2-x^2)^{-1} dx = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C,\ a^2 > x^2$$

$$\int (a+bx)^{-1} dx = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int x(x^2 \pm a^2)^{-\frac{1}{2}} dx = \sqrt{x^2 \pm a^2} + C$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{c}} \ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c>0$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{-c}} \operatorname{asin}\left[\frac{-2cx-b}{\sqrt{b^2-4ac}}\right] + C,\ c<0$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int x(a+bx)^{\frac{1}{2}}dx = -\frac{2(2a-3bx)(a+bx)^{\frac{3}{2}}}{15b^2}$$

$$\int (1+a^2x^2)^{\frac{1}{2}}dx = \frac{x(1+a^2x^2)^{\frac{1}{2}} + \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{a}}{2}$$

$$\int x(1+a^2x^2)^{\frac{1}{2}}dx = \frac{a\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}}}{3}$$

$$\int x^2(1+a^2x^2)^{\frac{1}{2}}dx = \frac{ax}{4}\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}} - \frac{8}{8a^2}x(1+a^2x^2)^{\frac{1}{2}} - \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{8a^3}$$

$$\int (1-a^2x^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x(1-a^2x^2)^{\frac{1}{2}} + \frac{\operatorname{asin}(ax)}{a}\right]$$

$$\int x(1-a^2x^2)^{\frac{1}{2}}dx = -\frac{a}{3}\left(\frac{1}{a^2}-x^2\right)^{\frac{3}{2}}$$

$$\int x^2(a^2-x^2)^{\frac{1}{2}}dx = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{8}\left[x(a^2-x^2)^{\frac{1}{2}} + a^2\operatorname{asin}\left(\frac{x}{a}\right)\right]$$

$$\int (1+a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\ln\left[x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right]$$

$$\int (1-a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\operatorname{asin}(ax) = -\frac{1}{a}\operatorname{acos}(ax)$$

• Integrals using the natural logarithm base ‘e’,

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

38.3.1 Integration Examples

• Integration by parts - It is normal to have to do the integration by parts more than once to solve a problem.

$$\int x^2 \sin 4x dx$$

$$u =$$

$$v =$$

• Substitution.

$$\int te^{t^2+1}dt$$

guess,

$$w = t^2 + 1 \qquad \frac{d}{dt}w = 2t \qquad dw = 2tdt$$

$$\int te^{t^2+1}dt = \int e^{t^2+1}tdt = \int e^w\frac{1}{2}dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{t^2+1} + C$$

• Partial fractions can be used to reduce complex polynomials to simple to integrate forms.

$$\int\left(\frac{5x+10}{x^2+3x+2}\right)dx =$$

38.4 Vectors

• When dealing with large and/or time varying objects or phenomenon we must be able to describe the state at locations, and as a whole. To do this vectors are a very useful tool.

• Consider a basic function and how it may be represented with partial derivatives.

$$y = f(x, y, z)$$

We can write this in differential form, but the right hand side must contain partial derivatives. If we separate the operators from the function, we get a simpler form. We can then look at them as the result of a dot product, and divide it into two vectors.

$$(d)y = \left(\left(\frac{\partial}{\partial x}\right)f(x, y, z)\right)dx + \left(\left(\frac{\partial}{\partial y}\right)f(x, y, z)\right)dy + \left(\left(\frac{\partial}{\partial z}\right)f(x, y, z)\right)dz$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\right)dx + \left(\frac{\partial}{\partial y}\right)dy + \left(\frac{\partial}{\partial z}\right)dz\right]f(x, y, z)$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \bullet (dx\underline{i} + dy\underline{j} + dz\underline{k})\right]f(x, y, z)$$

We then replace these vectors with the operators below. In this form we can manipulate the equation easily (whereas the previous form was very awkward).

$$(d)y = [\nabla \bullet dX]f(x, y, z)$$

$$(d)y = \nabla f(x, y, z) \bullet dX$$

$$(d)y = |\nabla f(x, y, z)||dX|\cos\theta$$

In summary,

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$F = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

$\nabla \bullet F =$ the divergence of function F

$\nabla \times F =$ the curl of function F

• Gauss's or Green's or divergence theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_V (\nabla \bullet F)dV = \oint_A F dA$$

where,

$V, A =$ a volume V enclosed by a surface area A

$F =$ a field or vector value over a volume

• Stoke's theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_A (\nabla \times F) dA = \oint_L F dL$$

where,

$A, L =$ A surface area A, with a bounding parimeter of length L

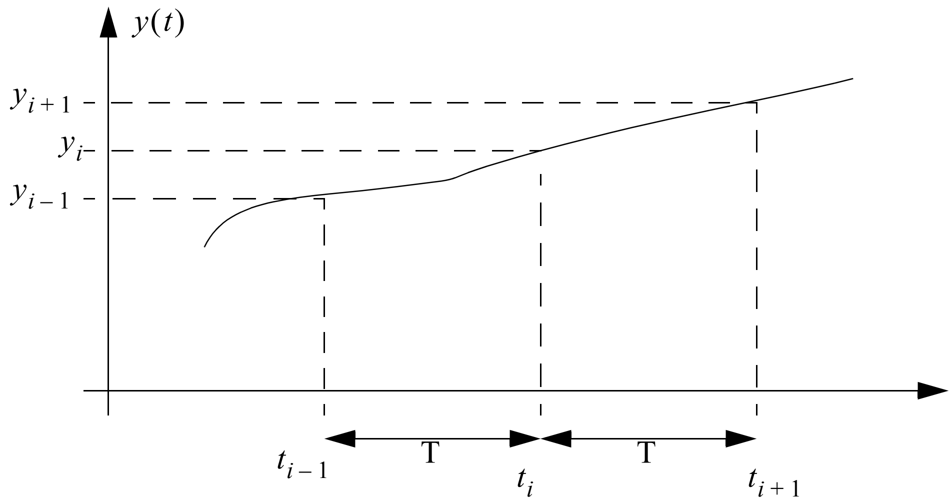
$F =$ a field or vector value over a volume

38.5 Numerical Tools

38.5.1 Approximation of Integrals and Derivatives from Sampled Data

- This form of integration is done numerically - this means by doing repeated calculations to solve the equation. Numerical techniques are not as elegant as solving differential equations, and will result in small errors. But these techniques make it possible to solve complex problems much faster.

- This method uses forward/backward differences to estimate derivatives or integrals from measured data.



$$\int_{t_{i-1}}^{t_i} y(t_i) \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

38.5.2 Centroids and Moments of Inertia

38.6 Problems

1. Find the following derivatives.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right)$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4))$

ans. $\frac{-1}{(x+1)^2}$

$-e^{-t}\sin(2t-4) + 2e^{-t}\cos(2t-4)$

2. Solve the following integrals.

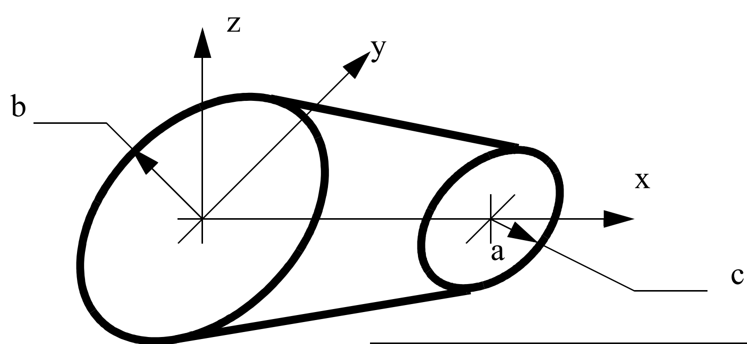
a) $\int e^{2t} dt$

b) $\int (\sin\theta + \cos 3\theta) d\theta$

ans. $\frac{1}{2}e^{2t} + C$

$-\cos\theta + \frac{1}{3}\sin 3\theta + C$

3. Set up an integral and solve it to find the area inside the volume below. The shape is basically a cone with the top cut off.



(ans. $V = \pi\left(\frac{c^2 a}{3} + \frac{b^2 a}{3} + \frac{abc}{3}\right)$

(move later) 4. Write a program that integrates the following function using the trapezoidal rule and Simpson's rule. The period of integration should range from 0 to 10 seconds. The program should compare the numerical results to the exact result.

$$y(t) = 5t^2 + \sin(20t)$$

```
// sample program
function foo = y(t)
        foo = 5 * t ^ 2 + sin ( 20 * t );
endfunction

sum = 0;
h = 0.1; // step size
for t = 0:h:10
        sum = sum + h * (f(t+h) + f(t)) / 2; // uses the trapezoid rule
end

t = 10;
actual = (5 / 3) * t ^ 3 - 0.05 * cos(20 * t);
mprintf("integral value numerical = %f, actual = %f \n", sum, actual);
// the value should be 1667 approximately
```

(move later - near end) 5. Write a program that finds the location of the minimum value for the function given below.

$$y(x) = \sin(\sin(5x^2) + \cos(20x) - 5x) + (x - 10)^2$$

```
// sample program
function foo = y(x)
        foo = sin(sin(5 * x * x) + cos(20 * x) - 5*x) + (x - 10) ^ 2;
endfunction

y_min = 100000000000; // something big to start
x_min = 0; // this value doesn't matter
h = 0.001; // step size - also the accuracy
for x = -100:h:100
        if (y(x) < y_min), // look for the smallest value
        x_min = x;
        y_min = y(x);
        end
end
x_min, y_min
```

8. Differentiate.

a)	$(x^3+4)^4$	(ans. $4(3)x^2(x^3+4)^3$
b)	$(x^3+4)^{-4}$	$-4(3)x^2(x^3+4)^{-5}$
c)	$\ln(x)$	$\frac{1}{x}$
d)	$\ln(x+x^3)$	$\frac{3x^2+1}{x^3+x}$
e)	e^x	e^x
f)	$e^{x+x^2-x^3}$	$(-3x^2+2x+1)e^{x+x^2-x^3}$
g)	$\frac{x}{x^2+5}$	$\frac{-x^2+5}{(x^2+5)^2}$
h)	$\frac{x^2}{x^3+5x}$	$\frac{-x^2+5}{(x^2+5)^2}$
i)	e^{x^2+5}	$2xe^{x^2+5}$
j)	$\sin(x^2)$	$2x\cos x^2$
k)	$e^{x^2}\sin(x)$	$2xe^{x^2}\sin(x)+e^{x^2}\cos(x)$
l)	$\frac{\sin x}{x}$	$\frac{x\cos x-\sin x}{x^2}$

9. Integrate the following indefinite functions with respect to x.

a) $\frac{1}{x}$

b) $\frac{x}{x+5}$

c) $\frac{1}{x^3}$

d) $5x^3 + 2x^2$

e) xe^{x^2}

f) $\cos x$

g) $(\cos x)^2$

(ans.

$\ln(x) + C$

$x - 5\ln(x+5) + C$

$\frac{-1}{2x^2} + C$

$1.25x^4 + \frac{2}{3}x^3 + C$

$\frac{1}{2}e^{x^2} + C$

$\sin x + C$

$\frac{\sin x \cos x + x}{2} + C$

10. Solve the following integrals.

a) $\int_0^{\infty} e^{-t} dt$

b) $\int_0^{10} (t^2 + 5t^3) dt$

c) $\int_0^{4\pi} \cos(t) dt$

(ans

-1

12833.33

0

11. Write a Scilab program that numerically intergrates the following functions.

a) $\int_0^{10} (t^2 + 5t^3) dt$

b) $\int_0^{4\pi} \cos(t) dt$

1. ANALYSIS OF DIFFERENTIAL EQUATIONS

Topics:
- First and second-order homogeneous differential equations
- Non-homogeneous differential equations
- First and second-order responses
- Non-linear system elements
- Design case

Objectives:
- To develop explicit equations that describe a system response.
- To recognize first and second-order equation forms.

1.1 Introduction

In the previous chapter we derived differential equations of motion for translating systems. These equations can be used to analyze the behavior of the system and make design decisions. The most basic method is to select a standard input type (a forcing function) and initial conditions, and then solve the differential equation. It is also possible to estimate the system response without solving the differential equation as will be discussed later.

Figure 1.1 shows an abstract description of a system. The basic concept is that the system changes the inputs to outputs. Say, for example, that the system to be analyzed is an elevator. Inputs to the system would be the mass of human riders and desired elevator height. The output response of the system would be the actual height of the elevator. For analysis, the system model could be developed using differential equations for the motor, elastic lift cable, mass of the car, etc. A basic test would involve assuming that the elevator starts at the ground floor and must travel to the top floor. Using assumed initial values and input functions the differential equation could be solved to get an explicit equation for elevator height. This output response can then be used as a guide to modify design choices (parameters). In practice, many of the assumptions and tests are mandated by law or by groups such as Underwriters Laboratories (UL), Canadian Standards Association (CSA) and the European Commission (CE).

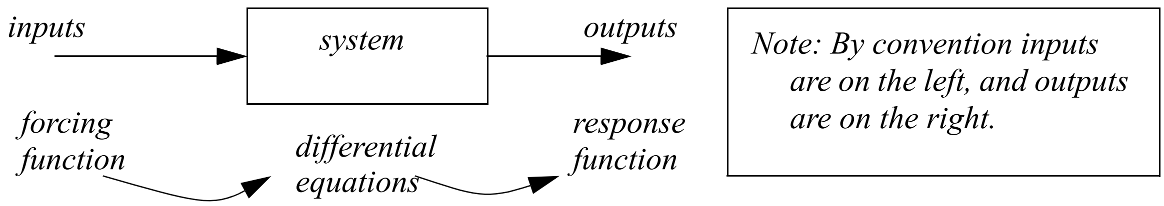


Figure 1.1 A system with and input and output response

There are several standard input types used to test a system. These are listed below in order of relative popularity with brief explanations.

- step - a sudden change of input, such as very rapidly changing a desired speed from 0Hz to 50Hz.
- ramp - a continuously increasing input, such as a motor speed that increases constantly at 10Hz per minute.
- sinusoidal - a cyclic input that varies continuously, such as wave height that is continually oscillating at 1Hz.
- parabolic - an exponentially increasing input, such as a motor speed that is 2Hz at 1 second, 4rad/s at 2 seconds, 8rad/s at 3 seconds, etc.

After the system has been modeled, an input type has been chosen, and the initial conditions have been selected, the system can be analyzed to determine its behavior. The most fundamental technique is to integrate the differential equation(s) for the system.

1.2 Explicit Solutions

Solving a differential equation results in an explicit solution. This equation provides the general response as a function of time, but it can also be used to find frequencies and other characteristics of interest. This section will review techniques used to integrate first and second-order homogenous differential equations. These equations correspond to systems without inputs, also called unforced systems. Non-homogeneous differential equations will also be reviewed.

The basic types of differential equations are shown in Figure 1.2. Each of these equations is linear. On the left hand side is the integration variable ’x’. If the right hand side is zero, then the equation is homogeneous. Each of these equations is linear because each of the terms on the left hand side is simply multiplied by a linear coefficient.

$$A\dot{x} + Bx = 0$$ *first-order homogeneous*

$$A\dot{x} + Bx = Cf(t)$$ *first-order non-homogeneous*

$$A\ddot{x} + B\dot{x} + Cx = 0$$ *second-order homogeneous*

$$A\ddot{x} + B\dot{x} + Cx = Df(t)$$ *second-order non-homogeneous*

Figure 1.2 Standard equation forms

A general solution for a first-order homogeneous differential equation is given in Figure 1.3. The solution begins with the solution of the homogeneous equation where a general form is 'guessed'. Substitution leads to finding the value of the coefficient 'Y'. Following this, the initial conditions for the equation are used to find the value of the coefficient 'X'. Notice that the final equation will begin at the initial displacement, but approach zero as time goes to infinity. The e-to-the-x behavior is characteristic for a first-order response.

Given the general form of a first-order homogeneous equation,

$$A\dot{x} + Bx = 0 \qquad \text{and} \qquad x(0) = x_0$$

initial condition

Guess a solution form and solve.

$$x = Xe^{-Yt} \qquad \dot{x} = -YXe^{-Yt}$$

$$A(-YXe^{-Yt}) + B(Xe^{-Yt}) = 0$$

$$A(-Y) + B = 0$$

$$Y = \frac{B}{A}$$

Therefore the general form is,

$$x_h = Xe^{-\frac{B}{A}t}$$

Next, use the initial conditions to find the remaining unknowns.

$$x_h = Xe^{-\frac{B}{A}t}$$

$$x_0 = Xe^{-\frac{B}{A}0}$$

$$x_0 = X$$

Therefore the final equation is,

$$x(t) = x_0 e^{-\frac{B}{A}t}$$

Note: The general form below is useful for finding almost all homogeneous equations

$$x_h(t) = Xe^{-Yt}$$

Figure 1.3 General solution of a first-order homogeneous equation

Solve the following differential equation given the initial condition.

$$\dot{x} + 2x = 0 \qquad x(0) = 3$$

ans. $x(t) = 3e^{-2t}$

Figure 1.4 Drill Problem: First order homogeneous differential equation

The general solution to a second-order homogeneous equation is shown in Figure 1.5. The solution begins with a guess of the homogeneous solution, and the solution of a quadratic equation. There are three possible cases that result from the solution of the quadratic equation: different but real roots; two identical real roots; or two complex roots. The three cases result in three different forms of solutions, as shown. The complex result is the most notable because it results in sinusoidal oscillations. It is not shown, but after the homogeneous solution has been found, the initial conditions need to be used to find the remaining coefficient values.

Given,

$$A\ddot{x} + B\dot{x} + Cx = 0 \qquad x(0) = x_0 \qquad and \qquad \dot{x}(0) = v_0$$

Guess a general equation form and substitute it into the differential equation,

$$x_h = Xe^{Yt} \qquad \dot{x}_h = YXe^{Yt} \qquad \ddot{x}_h = Y^2Xe^{Yt}$$

$$A(Y^2Xe^{Yt}) + B(YXe^{Yt}) + C(Xe^{Yt}) = 0$$

$$A(Y^2) + B(Y) + C = 0$$

$$Y = \frac{-B \pm \sqrt{(B)^2 - 4(AC)}}{2A} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note: There are three possible outcomes of finding the roots of the equations: two different real roots, two identical real roots, or two complex roots. Therefore there are three fundamentally different results.

If the values for Y are both real, but different, the general form is,

$$Y = R_1, R_2 \qquad x_h = X_1e^{R_1t} + X_2e^{R_2t}$$

Note: The initial conditions are then used to find the values for X_1 and X_2.

If the values for Y are both real, and identical, the general form is,

$$Y = R_1, R_1 \qquad x_h = X_1e^{R_1t} + X_2te^{R_1t}$$

The initial conditions are then used to find the values for X_1 and X_2.

If the values for Y are complex, the general form is,

$$Y = \sigma \pm \omega j \qquad x_h = X_3e^{\sigma t}\cos(\omega t + X_4)$$

The initial conditions are then used to find the values of X_3 and X_4.

Figure 1.5 Solution of a second-order homogeneous equation

As mentioned above, a complex solution when solving the homogeneous equation results in a sinusoidal oscillation, as proven in Figure 1.6. The most notable part of the solution is that there is both a frequency of oscillation and a phase shift. This form is very useful for analyzing the frequency response of a system, as will be seen in a later chapter.

Consider the situation where the results of a homogeneous solution are the complex conjugate pair.

$$Y = R \pm Cj$$

This gives the general result, as shown below:

$$x = X_1 e^{(R + Cj)t} + X_2 e^{(R - Cj)t}$$

$$x = X_1 e^{Rt} e^{Cjt} + X_2 e^{Rt} e^{-Cjt}$$

$$x = e^{Rt}\left(X_1 e^{Cjt} + X_2 e^{-Cjt}\right)$$

$$x = e^{Rt}(X_1(\cos(Ct) + j\sin(Ct)) + X_2(\cos(-Ct) + j\sin(-Ct)))$$

$$x = e^{Rt}(X_1(\cos(Ct) + j\sin(Ct)) + X_2(\cos((Ct) - j\sin(Ct))))$$

$$x = e^{Rt}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}{\sqrt{(X_1 + X_2)^2 + j^2(X_1 - X_2)^2}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\frac{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - \left(X_1^2 + -2X_1X_2 + X_2^2\right)}}{\sqrt{X_1^2 + 2X_1X_2 + X_2^2 - \left(X_1^2 + -2X_1X_2 + X_2^2\right)}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin($$

$$x = e^{Rt}\frac{\sqrt{4X_1X_2}}{\sqrt{4X_1X_2}}((X_1 + X_2)\cos(Ct) + j(X_1 - X_2)\sin(Ct))$$

$$x = e^{Rt}\sqrt{4X_1X_2}\left(\frac{(X_1 + X_2)}{\sqrt{4X_1X_2}}\cos(Ct) + j\frac{(X_1 - X_2)}{\sqrt{4X_1X_2}}\sin(Ct)\right)$$

$$x = e^{Rt}\sqrt{4X_1X_2}\cos\left(Ct + \operatorname{atan}\left(\frac{(X_1 - X_2)}{(X_1 + X_2)}\right)\right)$$

$$x = e^{Rt}X_3\cos(Ct + X_4)$$

frequency *phase shift*

where, $X_3 = \sqrt{4X_1X_2}$

$$X_4 = \operatorname{atan}\left(\frac{(X_1 - X_2)}{(X_1 + X_2)}\right)$$

Figure 1.6 Phase shift solution for a second-order homogeneous differential equation

Note: Occasionally a problem solution might consist of both a sine and cosine term with the same frequency. These should normally be combined to a single term with a phase shift as shown below.

Recall the double angle formula,

$$\sin(\omega t + \theta) = \sin\omega t\cos\theta + \sin\theta\cos\omega t$$

This can be written in a more common form,

$$A(\sin\omega t\cos\theta + \sin\theta\cos\omega t) = A\sin(\omega t + \theta)$$

$$A\cos\theta\sin\omega t + A\sin\theta\cos\omega t = A\sin(\omega t + \theta)$$

$$B\sin\omega t + C\cos\omega t = A\sin(\omega t + \theta)$$

where,

$$B = A\cos\theta$$
$$C = A\sin\theta$$

$$A = \frac{B}{\cos\theta} = \frac{C}{\sin\theta}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{C}{B}$$

$$\boxed{\theta = \operatorname{atan}\left(\frac{C}{B}\right)}$$

$$\boxed{A = \sqrt{B^2 + C^2}}$$

Consider the example,

$$3\sin 5t + 4\cos 5t = \sqrt{3^2 + 4^2}\sin\left(5t + \operatorname{atan}\left(\frac{4}{3}\right)\right) = 5\sin(5t + 0.927)$$

Figure 1.7 Phase shift solution form

Solve the following differential equation given the initial condition.

$$\ddot{x} + 2\dot{x} + x = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 2$$

ans. $x(t) = e^{-t} + 3te^{-t}$

Figure 1.8 Drill Problem: Second order homogeneous differential equation

The methods for solving non-homogeneous differential equations builds upon the methods used for the solution of homogeneous equations. This process adds a step to find the particular solution of the equation. An example of the solution of a first-order non-homogeneous equation is shown in Figure 1.9. To find the homogeneous solution the non-homogeneous part of the equation is set to zero. To find the particular solution the final form must be guessed. This is then substituted into the equation, and the values of the coefficients are found. Finally the homogeneous and particular solutions are added to get the final equation. The overall response of the system can be obtained by adding the homogeneous and particular parts. This is acceptable because the equations are linear, and the principle of superposition applies. The homogeneous equation deals with the response to initial conditions, and the particular solution deals with the response to forced inputs.

Generally,

$$A\dot{x} + Bx = Cf(t) \qquad\qquad x(0) = x_0$$

First, find the homogeneous solution as before, in Figure 1.3.

$$x_h = x_0 e^{-\frac{B}{A}t}$$

Next, guess the particular solution by looking at the form of 'f(t)'. This step is highly subjective, and if an incorrect guess is made, it will be unsolvable. When this happens, just make another guess and repeat the process. An example is given below. In the case below the guess should be similar to the exponential forcing function.

For example, if we are given

$$6\dot{x} + 2x = 5e^{4t}$$

A reasonable guess for the particular solution is,

$$x_p = C_1 e^{4t} \qquad \dot{x}_p = 4C_1 e^{4t}$$

Substitute these into the differential equation and solve for A.

$$6\left(4C_1 e^{4t}\right) + 2\left(C_1 e^{4t}\right) = 5e^{4t}$$

$$24C_1 + 2C_1 = 5 \qquad\qquad \therefore C_1 = \frac{5}{26}$$

Combine the particular and homogeneous solutions.

$$x = x_p + x_h = \frac{5}{26}e^{4t} + x_0 e^{-\frac{6}{2}t}$$

Figure 1.9 Solution of a first-order non-homogeneous equation

The method for finding a particular solution for a second-order non-homogeneous differential equation is shown in Figure 1.10. In this example the forcing function is sinusoidal, so the particular result should also be sinusoidal. The final result is converted into a phase shift form.

Generally,

$$A\ddot{x} + B\dot{x} + Cx = Df(t) \qquad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

1. Find the homogeneous solution as before.

$$x_h = X_3 e^{\sigma t} \cos(\omega t + X_4)$$
or
$$x_h = X_1 e^{\sigma t} + X_2 t e^{\sigma t}$$
or
$$x_h = X_1 e^{\sigma_1 t} + X_2 e^{\sigma_2 t}$$

2. Guess the particular solution by looking at the form of 'f(t)'. This step is highly subjective, and if an incorrect guess is made it will be unsolvable. When this happens, just make another guess and repeat the process. For the purpose of illustration an example is given below. In the case below it should be similar to the sine function.

For example, if we are given

$$2\ddot{x} + 6\dot{x} + 2x = 2\sin(3t + 4)$$

A reasonable guess is,

$$x_p = A\sin(3t) + B\cos(3t)$$
$$\dot{x}_p = 3A\cos(3t) - 3B\sin(3t)$$
$$\ddot{x}_p = -9A\sin(3t) - 9B\cos(3t)$$

Substitute these into the differential equation ans solve for A and B.

$$2(-9A\sin(3t) - 9B\cos(3t)) + 6(3A\cos(3t) - 3B\sin(3t)) + 2(A\sin(3t) + B\cos(3t)) = 2\sin(3t+4)$$

$$(-18A - 18B + 2A)\sin(3t) + (-18B + 18A + 2B)\cos(3t) = 2\sin(3t + 4)$$

$$(-16A - 18B)\sin(3t) + (18A - 16B)\cos(3t) = 2(\sin 3t \cos 4 + \cos 3t \sin 4)$$

$$(-16A - 18B)\sin(3t) + (18A - 16B)\cos(3t) = (2\cos 4)\sin(3t) + (2\sin 4)\cos(3t)$$

$$-16A - 18B = 2\cos 4 \qquad 18A - 16B = 2\sin 4$$

$$\begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2\cos 4 \\ 2\sin 4 \end{bmatrix} \quad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -16 & -18 \\ 18 & -16 \end{bmatrix}^{-1} \begin{bmatrix} -1.307 \\ -1.514 \end{bmatrix} = \begin{bmatrix} -0.0109 \\ 0.0823 \end{bmatrix}$$

Next, rearrange the equation to phase shift form.

$$x_p = -0.0109\sin(3t) + 0.0823\cos(3t)$$

$$x_p = \sqrt{-0.0109^2 + 0.0823^2}\,\sin\left(3t + \operatorname{atan}\left(\frac{0.0823}{-0.0109}\right) + \frac{\pi}{2}\right)$$

3. Use the initial conditions to determine the coefficients in the homogeneous solution.

Figure 1.10 Solution of a second-order non-homogeneous equation

When guessing particular solutions, the forms in Figure 1.11 can be helpful.

Forcing Function	*Guess*
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} or Cxe^{Ax}
$B\sin(Ax)$ or $B\cos(Ax)$	$C\sin(Ax) + D\cos(Ax)$ or $Cx\sin(Ax) + xD\cos(Ax)$

Figure 1.11 General forms for particular solutions

Solve the following differential equation given the initial condition.

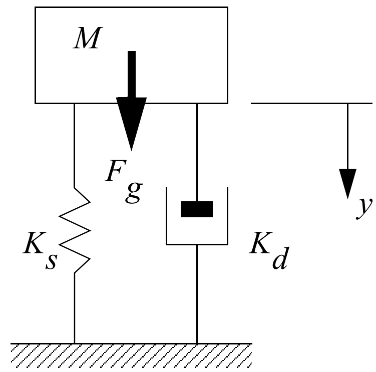
$$\ddot{x} + 2\dot{x} + x = 1 \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

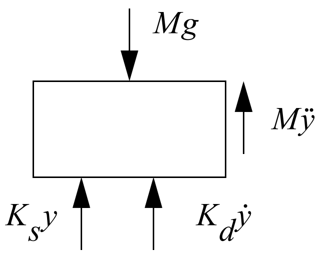
ans. $x(t) = -e^{-t} - te^{-t} + 1$

Figure 1.12 Drill Problem: Second order non-homogeneous differential equation

An example of a second-order system is shown in Figure 1.13. As expected, it begins with a FBD and summation of forces. This is followed with the general solution of the homogeneous equation. Real roots are assumed thus allowing the problem solution to continue in Figure 1.14.

Assume the system illustrated to the right starts from rest at a height 'h'. At time 't=0' the system is released and allowed to move.





$$+\uparrow \sum F_y = -Mg + K_s y + K_d \dot{y} = -M\ddot{y}$$

$$M\ddot{y} + K_d \dot{y} + K_s y = Mg$$

Find the homogeneous solution.

$$y_h = e^{At} \qquad \dot{y}_h = Ae^{At} \qquad \ddot{y}_h = A^2 e^{At}$$

$$M\ddot{y} + K_d \dot{y} + K_s y = 0$$

$$M\left(A^2 e^{At}\right) + K_d(Ae^{At}) + K_s(e^{At}) = 0$$

$$MA^2 + K_d A + K_s = 0$$

$$A = \frac{-K_d \pm \sqrt{K_d^2 - 4MK_s}}{2M}$$

Let us assume that the values of M, K_d and K_s lead to the case of two different positive roots. This would occur if the damper value was much larger than the spring and mass values. Thus,

$$A = R_1, R_2$$

$$y_h = C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

Figure 1.13 Second-order system example

The solution continues by assuming a particular solution and calculating values for the coefficients using the initial conditions in Figure 1.14. The final result is a second-order system that is overdamped, with no oscillation.

Next, find the particular solution.

$$y_p = C \qquad \dot{y}_h = 0 \qquad \ddot{y}_h = 0$$

$$M(0) + K_d(0) + K_s(C) = Mg$$

$$C = \frac{Mg}{K_s}$$

Now, add the homogeneous and particular solutions and solve for the unknowns using the initial conditions.

$$y(t) = y_p + y_h = \frac{Mg}{K_s} + C_1 e^{R_1 t} + C_2 e^{R_2 t}$$

$$y(0) = h \qquad y'(0) = 0$$

$$h = \frac{Mg}{K_s} + C_1 e^0 + C_2 e^0$$

$$C_1 + C_2 = h - \frac{Mg}{K_s}$$

$$y'(t) = R_1 C_1 e^{R_1 t} + R_2 C_2 e^{R_2 t}$$

$$0 = R_1 C_1 e^0 + R_2 C_2 e^0$$

$$0 = R_1 C_1 + R_2 C_2 \qquad C_1 = \frac{-R_2}{R_1} C_2$$

$$-\frac{R_2}{R_1} C_2 + C_2 = h - \frac{Mg}{K_s}$$

$$C_2 = \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) \qquad C_1 = \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right)$$

Now, combine the solutions and solve for the unknowns using the initial conditions.

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \frac{-R_2}{R_1}\left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_2 t}$$

$$y(t) = \frac{Mg}{K_s} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{-R_1}{R_1 - R_2}\right) e^{R_1 t} + \left(\frac{K_s h - Mg}{K_s}\right)\left(\frac{R_2}{R_1 - R_2}\right) e^{R_2 t}$$

Figure 1.14 Second-order system example (continued)

Given,

$$A \sin(\omega t + \theta) \qquad \textit{(the desired final form)}$$

$$A(\cos \omega t \sin \theta + \sin \omega t \cos \theta)$$

$$(A \sin \theta) \cos \omega t + (A \cos \theta) \sin \omega t$$

$$B \cos \omega t + C \sin \omega t \qquad \textit{(the form we start with)}$$

where,

$$B = A \sin \theta$$

$$C = A \cos \theta$$

To find theta,

$$\frac{B}{C} = \frac{A \sin \theta}{A \cos \theta} = \tan \theta$$

$$\theta = \operatorname{atan}\left(\frac{B}{C}\right)$$

To find A, (method #1)

$$A = \frac{B}{\sin \theta} = \frac{C}{\cos \theta}$$

To find A, (method #2)

$$A = \sqrt{B^2 + C^2}$$

For example,

$$3 \cos 5t + 4 \sin 5t$$

$$\sqrt{3^2 + 4^2} \sin\left(5t + \operatorname{atan}\frac{3}{4}\right)$$

$$5 \sin(5t + 0.6435)$$

Figure 1.15 Proof for conversion to phase form

1.3 Responses

Solving differential equations tends to yield one of two basic equation forms. The e-to-the-negative-t forms are the first-order responses and slowly decay over time. They never naturally oscillate, and only oscillate if forced to do so. The second-order forms may include natural oscillation.

1.3.1 First-order

A first-order system is described with a first-order differential equation. The response function for these systems is natural decay or growth as shown in Figure 1.16. The time constant for the system can be found directly from the differential equation. It is a measure of how quickly the system responds to a change. When an input to a system has changed, the system output will be approximately 63% of the way to its final value when the elapsed time equals the time constant. The initial and final values of the function can be determined algebraically to find the first-order response with little effort.

If we have experimental results for a system, we can calculate the time constant, initial and final values. The time constant can be found two ways, one by extending the slope of the first part of the curve until it intersects the final value line. That time at the intersection is the time constant. The other method is to look for the time when the output value has shifted 63.2% of the way from the initial to final values for the system. Assuming the change started at t=0, the time at this point corresponds to the time constant.

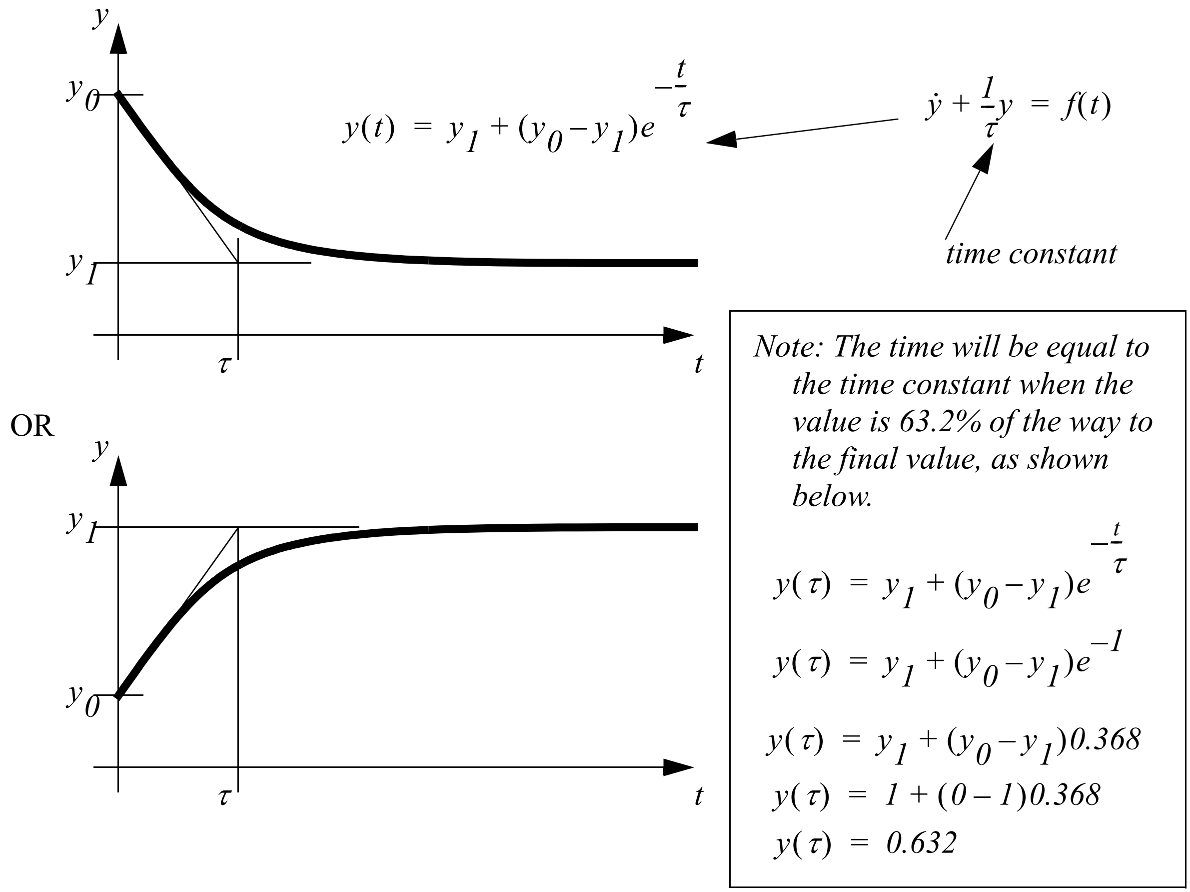
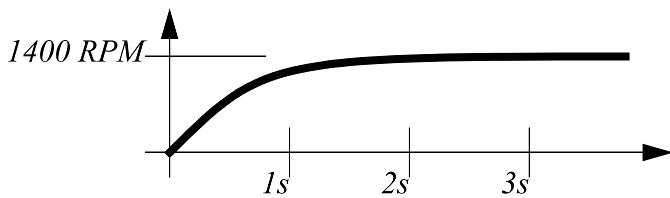


Figure 1.16 Typical first-order responses

The example in Figure 1.17 calculates the coefficients for a first-order differential equation given a graphical output response to an input. The differential equation is for a permanent magnet DC motor, and will be examined in a later chapter. If we consider the steady state when the speed is steady at 1400RPM, the first derivative will be zero. This simplifies the equation and allows us to calculate a value for the parameter K in the differential equation. The time constant can be found by drawing a line asymptotic to the start of the motor curve, and finding the point where it intercepts the steady-state value. This gives an approximate time constant of 0.8 s. This can then be used to calculate the remaining coefficient. Some additional numerical calculation leads to the final differential equation as shown.

For the motor, use the differential equation and the speed curve when Vs=10V is applied:



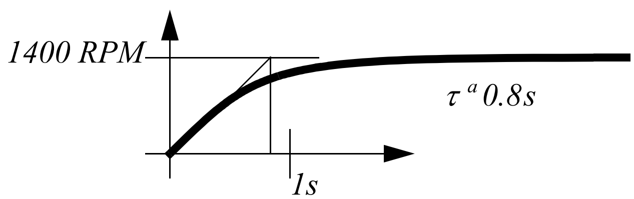
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$



$$\tau \approx 0.8s$$

$$\left(\frac{K^2}{JR}\right) = \frac{1}{0.8s}$$

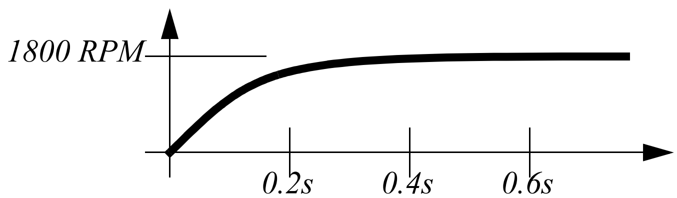
$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

$$\frac{K}{JR} = 18.328$$

$$\dot{\omega} + \frac{1}{0.8}\omega = 18.328V_s$$

Figure 1.17 Finding an equation using experimental data

Find the differential equation when a step input of Vs=12V is applied:



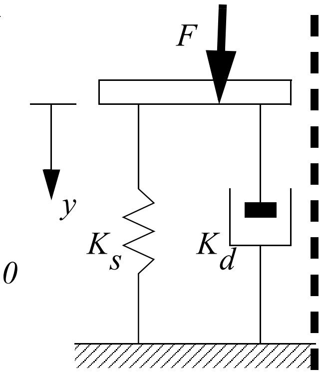
$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$

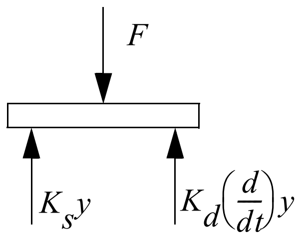
ans. $$\left(\frac{d}{dt}\right)\omega + \frac{1}{0.15}\omega = \frac{1800}{12(0.15)}V_s$$

Figure 1.18 Drill problem: Find the constants for the equation

A simple mechanical example is given in Figure 1.19. The modeling starts with a FBD and a sum of forces. After this, the homogenous solution is found by setting the non-homogeneous part to zero and solving. Next, the particular solution is found, and the two solutions are combined. The initial conditions are used to find the remaining unknown coefficients.

Find the response to the applied force if the force is applied at t=0s. Assume the system is initially deflected a height of h.





$$+\uparrow \sum F_y = -F + K_s y + K_d\left(\frac{d}{dt}\right)y = 0$$

$$K_d\dot{y} + K_s y = F$$

Find the homogeneous solution.

$$y_h = Ae^{Bt} \qquad \dot{y}_h = ABe^{Bt}$$

$$K_d(ABe^{Bt}) + K_s(Ae^{Bt}) = 0$$

$$K_d B + K_s = 0$$

$$B = \frac{-K_s}{K_d}$$

Next, find the particular solution.

$$y_p = C \qquad \dot{y}_p = 0$$

$$K_d(0) + K_s(C) = F \qquad \therefore C = \frac{F}{K_s}$$

Combine the solutions, and find the remaining unknown.

$$y(t) = y_p + y_h = Ae^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

$$y(0) = h$$

$$h = Ae^{0} + \frac{F}{K_s} \qquad \therefore A = h - \frac{F}{K_s}$$

The final solution is,

$$y(t) = \left(h - \frac{F}{K_s}\right)e^{\frac{-K_s}{K_d}t} + \frac{F}{K_s}$$

Figure 1.19 First-order system analysis example

Use the general form given below to solve the problem in Figure 1.19 without solving the differential equation. Assume the system starts at y=-20.

$$\dot{y} + 10y = 5 \longrightarrow y(t) = y_1 + (y_0 - y_1)e^{-\frac{t}{\tau}}$$

ans. $y(t) = 0.5 - 19.5e^{-10t}$

Figure 1.20 Drill problem: Developing the final equation using the first-order model form

A first-order system tends to be passive, meaning it doesn't deliver energy or power. A first-order system will not oscillate unless the input forcing function is also oscillating. The output response lags the input and the delay is determined by the system's time constant.

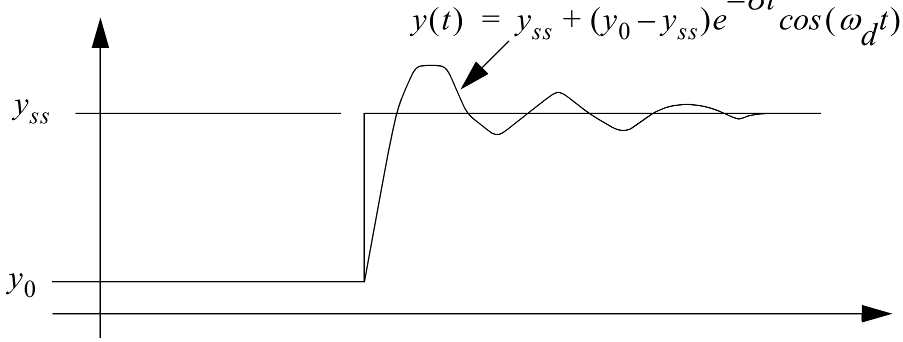
1.3.2 Second-order

A second-order system response typically contains two first-order responses, or a first-order response and a sinusoidal component. A typical sinusoidal second-order response is shown in Figure 1.21. Notice that the coefficients of the differential equation include a damping coefficient and a natural frequency. These can be used to develop the final response, given the initial conditions and forcing function. Notice that the damped frequency of oscillation is the actual frequency of oscillation. The damped frequency will be lower than the natural frequency when the damping coefficient is between 0 and 1. If the damping coefficient is greater than one the damped frequency becomes negative, and the system will not oscillate because it is overdamped.

A second-order system, and a typical response to a stepped input.

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f(t) \longrightarrow \sigma = \zeta\omega_n \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$y(t) = y_{ss} + (y_0 - y_{ss})e^{-\sigma t}\cos(\omega_d t)$$



ω_n *Natural frequency of system - Approximate frequency of control system oscillations.*

ξ *Damping factor of system - If < 1 underdamped, and system will oscillate. If =1 critically damped. If > 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.*

ω_d *The actual frequency of oscillation - It is below the natural frequency because of the damping.*

Figure 1.21 The general form for a second-order system

When only the damping coefficient is increased, the frequency of oscillation, and overall response time will slow, as seen in Figure 1.22. When the damping coefficient is 0 the system will oscillate indefinitely. Critical damping occurs when the damping coefficient is 1. At this point both roots of the differential equation are equal. The system will not oscillate if the damping coefficient is greater than or equal to 1.

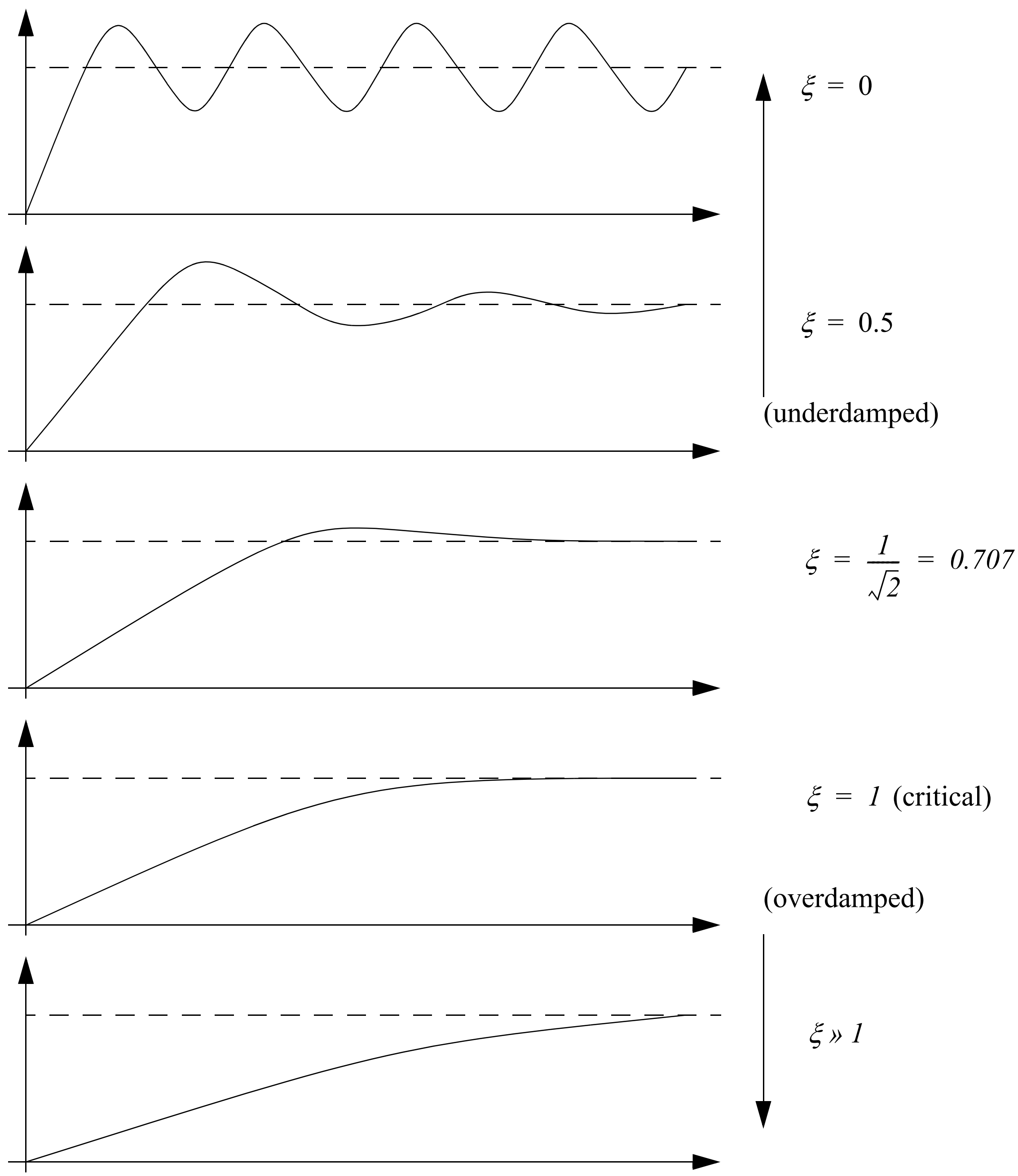


Figure 1.22 The effect of the damping coefficient

When observing second-order systems it is more common to use more direct measurements of the response. Some of these measures are shown in Figure 1.23. The rise time is the time it takes to go from 10% to 90% of the total displacement, and is comparable to a first order time constant. The settling time indicates how long it takes for the system to pass within a tolerance band around the final value. The permissible zone shown is 2%, but if it were larger the system would have a shorter settling time. The period of oscillation can be measured directly as the time between peaks of the oscillation, the inverse is the damping frequency. (Note: don't forget to convert to radians.) The damped frequency can also be found using the time to the first peak, as half the period. The overshoot is the height of the first peak. Using the time to the first peak, and the overshoot the damping coefficient can be found.

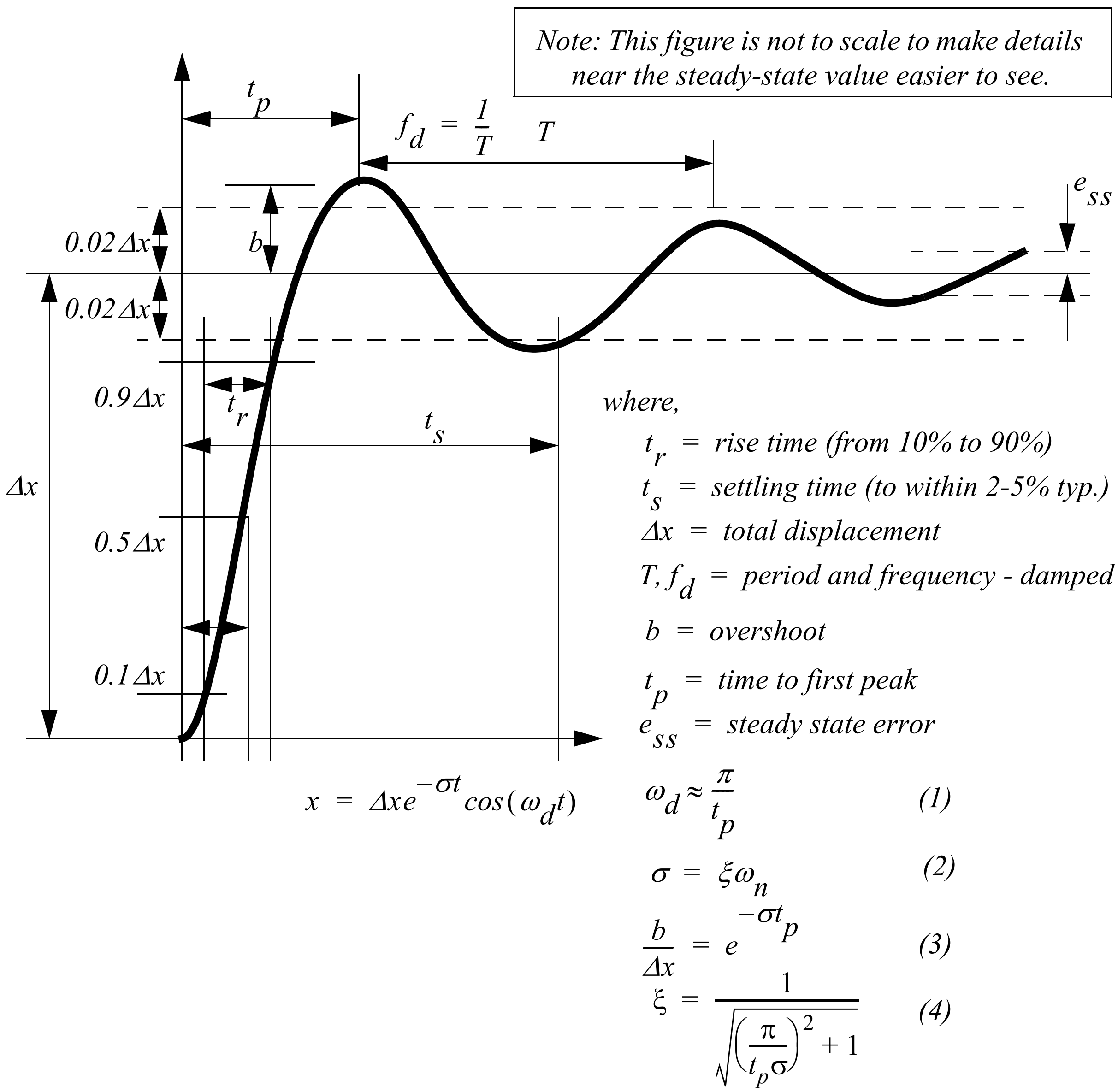


Figure 1.23 Characterizing a second-order response (not to scale)

Note: We can calculate these relationships using the complex homogenous form, and the generic second order equation form.

$$A^2 + 2\xi\omega_n A + \omega_n^2 = 0$$

$$A = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \sigma \pm j\omega_d$$

$$\frac{-2\xi\omega_n}{2} = \sigma = -\xi\omega_n \qquad \omega_n = \frac{\sigma}{-\xi} \qquad (1)$$

$$\frac{\sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = j\omega_d$$

$$4\xi^2\omega_n^2 - 4\omega_n^2 = 4(-1)\omega_d^2$$

$$\omega_n^2 - \xi^2\omega_n^2 = \omega_d^2 \qquad \omega_n\sqrt{1-\xi^2} = \omega_d \qquad (2)$$

$$\frac{\sigma^2}{\xi^2} - \xi^2\frac{\sigma^2}{\xi^2} = \omega_d^2$$

$$\frac{1}{\xi^2} = \frac{\omega_d^2}{\sigma^2} + 1 \qquad \xi = \frac{1}{\sqrt{\frac{\omega_d^2}{\sigma^2} + 1}} \qquad (3)$$

The time to the first peak can be used to find the approximate decay constant

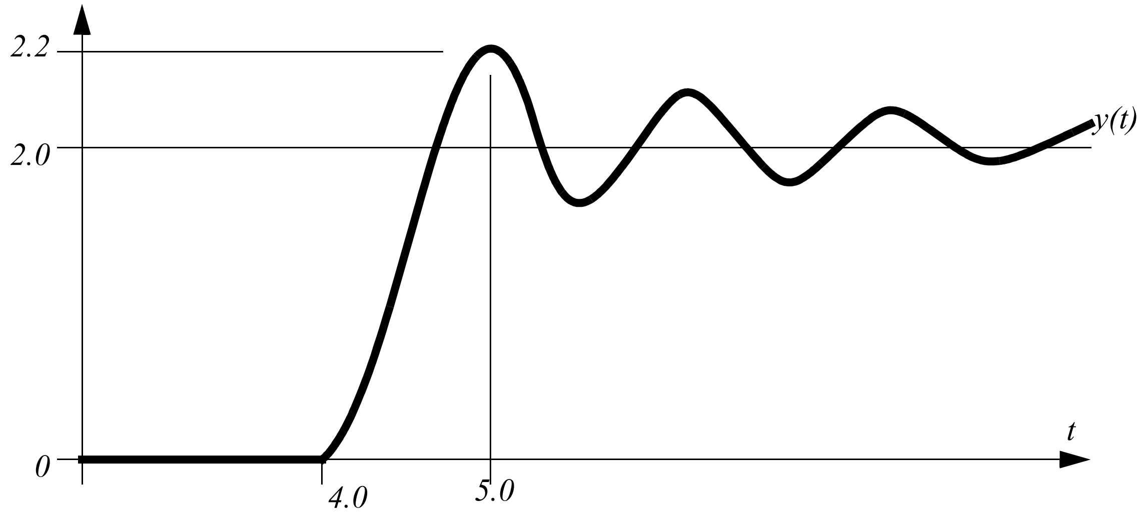
$$x(t) = C_1 e^{-\sigma t}\cos(\omega_d t + C_2)$$

$$\omega_d = \frac{\pi}{t_p} \qquad (4)$$

$$b \approx \Delta x e^{-\sigma t_p}(1)$$

$$\sigma = -\frac{\ln\left(\frac{b}{\Delta x}\right)}{t_p} \qquad (5)$$

Figure 1.24 Second order relationships between damped and natural frequency



Write a function of time for the graph. (Note: measure, using a ruler, to get values.) Find the natural frequency and damping coefficient to develop the differential equation. Using the dashed lines determine the settling time.

ans.

$t < 4 \qquad y(t) = 0$

$t \geq 4 \qquad y(t) =$

Figure 1.25 Drill problem: Find the equation given the response curve

1.3.3 Other Responses

First-order systems have e-to-the-t type responses. Second-order systems add another e-to-the-t response or a sinusoidal excitation. As we move to higher order linear systems we typically add more e-to-the-t terms, and/or more sinusoidal terms. A possible higher order system response is seen in Figure 1.26. The underlying function is a first-order response that drops at the beginning, but levels out. There are two sinusoidal functions superimposed, one with about one period showing, the other with a much higher frequency.

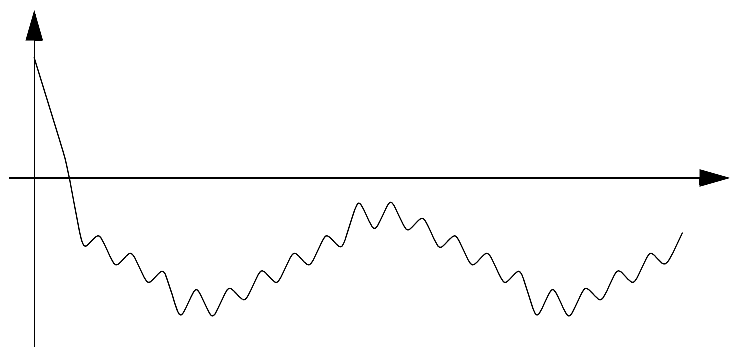


Figure 1.26 An example of a higher order system response

The basic techniques used for solving first and second-order differential equations can be applied to higher order differential equations, although the solutions will start to become complicated for systems with much higher orders. The example in Figure 1.27 shows a fourth order differential equation. In this case the resulting homogeneous solution yields four roots. The result in this case are two real roots, and a complex pair. The two real roots result in e-to-the-t terms, while the complex pair results in a damped sinusoid. The particular solution is relatively simple to find in this example because the non-homogeneous term is a constant.

Given the homogeneous differential equation

$$\left(\frac{d}{dt}\right)^4 x + 13\left(\frac{d}{dt}\right)^3 x + 34\left(\frac{d}{dt}\right)^2 x + 42\left(\frac{d}{dt}\right)x + 20x = 5$$

Guess a solution for the homogeneous equation,

$$x_h = e^{At}$$

$$\frac{d}{dt}x_h = Ae^{At} \quad \left(\frac{d}{dt}\right)^2 x_h = A^2 e^{At} \quad \left(\frac{d}{dt}\right)^3 x_h = A^3 e^{At} \quad \left(\frac{d}{dt}\right)^4 x_h = A^4 e^{At}$$

Substitute the values into the differential equation and find a value for the unknown.

$$A^4 e^{At} + 13A^3 e^{At} + 34A^2 e^{At} + 42Ae^{At} + 20e^{At} = 0$$

$$A^4 + 13A^3 + 34A^2 + 42A + 20 = 0$$

$$A = -1, -10, -1-j, -1+j$$

$$x_h = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t} \cos(t + C_4)$$

Guess a particular solution, and the solve for the coefficient.

$$x_p = A \quad \frac{d}{dt}x_p = 0 \quad \left(\frac{d}{dt}\right)^2 x_p = 0 \quad \left(\frac{d}{dt}\right)^3 x_p = 0 \quad \left(\frac{d}{dt}\right)^4 x_p = 0$$

$$0 + 13(0) + 34(0) + 42(0) + 20A = 5 \qquad A = 0.25$$

Figure 1.27 Solution of a higher order differential equation

The example is continued in Figure 1.28 and Figure 1.29 where the initial conditions are used to find values for the coefficients in the homogeneous solution.

Solve for the unknowns, assuming the system starts at rest and undeflected.

$$x(t) = C_1 e^{-t} + C_2 e^{-10t} + C_3 e^{-t} cos(t + C_4) + 0.25$$

$$0 = C_1 + C_2 + C_3 cos(C_4) + 0.25 \qquad (1)$$

$$C_3 cos(C_4) = -C_1 - C_2 - 0.25 \qquad (2)$$

$$\frac{d}{dt} x_h(t) = -C_1 e^{-t} - 10C_2 e^{-10t} - C_3 e^{-t} cos(t + C_4) - C_3 e^{-t} sin(t + C_4)$$

$$0 = -C_1 - 10C_2 - C_3 cos(C_4) - C_3 sin(C_4) \qquad (3)$$

Equations (1) and (3) can be added to get the simplified equation below.

$$0 = -9C_2 - C_3 sin(C_4) + 0.25$$

$$C_3 sin(C_4) = -9C_2 + 0.25 \qquad (4)$$

$$\left(\frac{d}{dt}\right)^2 x_h(t) = C_1 e^{-t} + 100C_2 e^{-10t} + C_3 e^{-t} cos(t + C_4) + C_3 e^{-t} sin(t + C_4) + C_3 e^{-t} sin(t + C_4) - C_3 e^{-t} cos(t + C_4)$$

$$0 = C_1 + 100C_2 + C_3 cos(C_4) + C_3 sin(C_4) + C_3 sin(C_4) - C_3 cos(C_4)$$

$$0 = C_1 + 100C_2 + 2C_3 sin(C_4) \qquad (5)$$

Equations (4) and (5) can be combined.

$$0 = C_1 + 100C_2 + 2(-9C_2 + 0.25)$$

$$0 = -17C_1 + 100C_2 + 0.5 \qquad (6)$$

$$\left(\frac{d}{dt}\right)^3 x_h(t) = -C_1 e^{-t} + (-1000)C_2 e^{-10t} - 2C_3 e^{-t} sin(t + C_4) + 2C_3 e^{-t} cos(t + C_4)$$

$$0 = -C_1 + (-1000)C_2 - 2C_3 \sin(C_4) + 2C_3 \cos(C_4) \qquad (7)$$

Figure 1.28 Solution of a higher order differential equation

Equations (2 and (4) are substituted into equation (7).

$$0 = -C_1 + (-1000)C_2 - 2(-9C_2 + 0.25) + 2(-C_1 - C_2 - 0.25)$$

$$0 = -3C_1 + (-984)C_2 - 1$$

$$C_1 = \left(-\frac{984}{3}\right)C_2 - \frac{1}{3} \qquad (8)$$

Equations (6) and (8) can be combined.

$$0 = -17\left(\left(-\frac{984}{3}\right)C_2 - \frac{1}{3}\right) + 100C_2 + 0.5$$

$$0 = 5676C_2 + 6.1666667 \qquad C_2 = -0.00109$$

$$C_1 = \left(-\frac{984}{3}\right)(-0.00109) - \frac{1}{3} \qquad C_1 = 0.0242$$

Equations (2) and (4) can be combined.

$$\frac{C_3 sin(C_4)}{C_3 cos(C_4)} = \frac{-9C_2 + 0.25}{-C_1 - C_2 - 0.25}$$

$$tan(C_4) = \frac{-9(-0.00109) + 0.25}{-(0.0242) - (-0.00109) - 0.25} \qquad C_4 = -0.760$$

Equation (4) can be used the find the remaining unknown.

$$C_3 sin(-0.760) = -9(-0.00109) + 0.25 \qquad C_3 = -0.377$$

The final response function is,

$$x(t) = 0.0242e^{-t} + (-0.00109)e^{-10t} + (-0.377)e^{-t}cos(t - 0.760) + 0.25$$

Figure 1.29 Solution of a higher order differential equation (cont'd)

In some cases we will have systems with multiple differential equations, or non-linear terms. In these cases explicit analysis of the equations may not be feasible. In these cases we may use other techniques, such as numerical integration, which will be covered in later chapters.

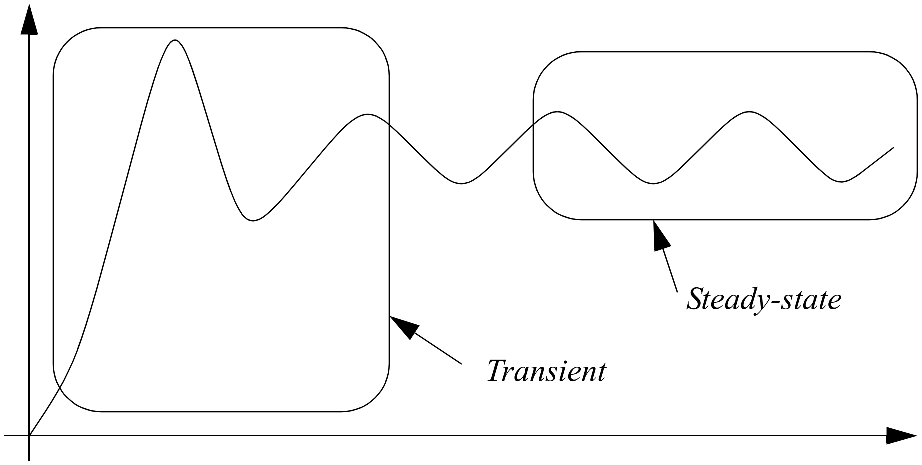
1.4 Response Analysis

Up to this point we have mostly discussed the process of calculating the system response. As an engineer, obtaining the response is important, but evaluating the results is more important. The most critical design consideration is system stability. In most cases a system should be inherently stable in all situations, such as a car "cruise control". In other cases an unstable system may be the objective, such as an explosive device. Simple methods for determining the stability of a system are listed below:

1. If a step input causes the system to go to infinity, it will be inherently unstable.
2. A ramp input might cause the system to go to infinity; if this is the case, the system might not respond well to constant change.
3. If the response to a sinusoidal input grows with each cycle, the system is probably resonating, and will become unstable.

Beyond establishing the stability of a system, we must also consider general performance. This includes the time constant for a first-order system, or damping coefficient and natural frequency for a second-order system. For example, assume we have designed an elevator that is a second-order system. If it is under damped the elevator will oscillate, possibly leading to motion sickness, or worse. If the elevator is over damped it will take longer to get to floors. If it is critically damped it will reach the floors quickly, without overshoot.

Engineers distinguish between initial setting effects (transient) and long term effects (steady-state). The transient effects are closely related to the homogeneous solution to the differential equations and the initial conditions. The steady-state effects occur after some period of time when the system is acting in a repeatable or non-changing form. Figure 1.30 shows a system response. The transient effects at the beginning include a quick rise time and an overshoot. The steady-state response settles down to a constant amplitude sine wave.



Note: the transient response is predicted with the homogeneous solution. The steady state response in mainly predicted with the particular solution, although in some cases the homogeneous solution might have steady state effects, such as a non-decaying oscillation.

Figure 1.30 A system response with transient and steady-state effects

1.5 Non-Linear Systems

Non-linear systems cannot be described with a linear differential equation. A basic linear differential equation has coefficients that are constant, and the derivatives are all first order. Examples of non-linear differential equations are shown in Figure 1.31.

$$\dot{x} + x^2 = 5$$

$$\dot{x}^2 + x = 5$$

$$\dot{x} + \log(x) = 5$$

$$\dot{x} + (5t)x = 5$$

Note: the sources of non-linearity are circled.

Figure 1.31 Examples of non-linear differential equations

Examples of system conditions that lead to non-linear solutions are,

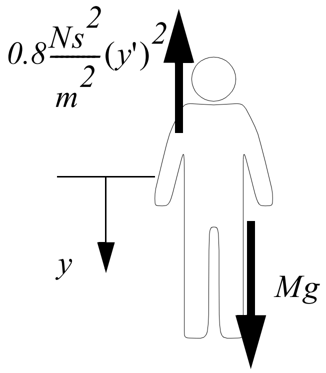
- aerodynamic drag
- forces that are a squared function of distance
- devices with non-linear responses

Explicitly solving non-linear differential equations can be difficult, and will typically involve complex solutions for simple problems.

1.5.1 Non-Linear Differential Equations

A non-linear differential equation is presented in Figure 1.32. It involves a person ejected from an aircraft with a drag force coefficient of 0.8. (Note: This coefficient is calculated using the drag coefficient and other properties such as the speed of sound and cross sectional area.) The FBD shows the sum of forces, and the resulting differential equation. The velocity squared term makes the equation non-linear, and so it cannot be analyzed with the previous methods. In this case the terminal velocity is calculated by setting the acceleration to zero. This results in a maximum speed of 126 kph.

Consider the differential equation for a 100kg human ejected from an airplane. The aerodynamic drag will introduce a squared variable, therefore making the equation non-linear.



$$\sum F_y = 0.8(\dot{y})^2 - Mg = -M\ddot{y}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 100kg 9.81\frac{N}{kg}$$

$$100kg\ddot{y} + 0.8\frac{Ns^2}{m^2}(\dot{y})^2 = 981N$$

$$100kg\ddot{y} + 0.8kg\frac{m}{s^2}\frac{s^2}{m^2}(\dot{y})^2 = 981kg\frac{m}{s^2}$$

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$\ddot{y} + 8 \times 10^{-3}m^{-1}(\dot{y})^2 = 9.81ms^{-2}$$

The terminal velocity can be found be setting the acceleration to zero.

$$(0) + 8 \times 10^{-3}m^{-1}(\dot{y})^2 = 9.81ms^{-2}$$

$$\dot{y} = \sqrt{\frac{9.81ms^{-2}}{8 \times 10^{-3}m^{-1}}} = \sqrt{\frac{9.81}{8 \times 10^{-3}}m^2s^{-2}} = 35.0\frac{m}{s} = 126\frac{km}{h}$$

Figure 1.32 Development of a non-linear differential equation

The equation can also be solved using explicit integration, as shown in Figure 1.33. In this case the equation is separated and rearranged to isolate the ’v’ terms on the left, and time on the right. The term is then integrated in Figure 1.34 and Figure 1.35. The final form of the equation is non-trivial, but contains e-to-t terms, as we would expect.

An explicit solution can begin by replacing the position variable with a velocity variable and rewriting the equation as a separable differential equation.

$$100\ddot{y} + 0.8m^{-1}(\dot{y})^2 = 981ms^{-2}$$

$$100\dot{v} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} + 0.8m^{-1}v^2 = 981ms^{-2}$$

$$100\frac{dv}{dt} = 981ms^{-2} - 0.8m^{-1}v^2$$

$$\frac{100}{981ms^{-2} - 0.8m^{-1}v^2}dv = dt$$

$$\int \frac{\frac{100}{-0.8m^{-1}}}{\frac{981}{-0.8m^{-1}}ms^{-2} + v^2}dv = \int dt$$

$$\int \frac{-125m}{v^2 - 1226.25m^2s^{-2}}dv = t + C_1$$

$$\int \frac{-125m}{\left(v + 35.02\frac{m}{s}\right)\left(v - 35.02\frac{m}{s}\right)}dv = t + C_1$$

Figure 1.33 Developing an integral

This can be reduced with a partial fraction expansion.

$$\int\left[\frac{A}{\left(v + 35.02\frac{m}{s}\right)} + \frac{B}{\left(v - 35.02\frac{m}{s}\right)}\right]dv = t + C_1$$

$$Av - A\left(35.02\frac{m}{s}\right) + Bv + B\left(35.02\frac{m}{s}\right) = -125m$$

$$v(A + B) + 35.02\frac{m}{s}(-A + B) = -125m$$

$$A + B = 0 \qquad\qquad A = -B$$

$$35.02\frac{m}{s}(-A + B) = -125m$$

$$(-(-B) + B) = -\frac{125}{35.02}s \qquad\qquad B = -1.785s$$

$$A = 1.785s$$

$$\int\left[\frac{1.785s}{\left(v + 35.02\frac{m}{s}\right)} + \frac{-1.785s}{\left(v - 35.02\frac{m}{s}\right)}\right]dv = t + C_1$$

The integral can then be solved using an identity from the integral table. In this case the integration constants can be left off because they are redundant with the one on the right hand side.

$$1.785s\,ln\left|v + 35.02\frac{m}{s}\right| - 1.785s\,ln\left|v - 35.02\frac{m}{s}\right| = t + C_1$$

$$1.785s\,ln\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = t + C_1$$

$$\boxed{\int(a + bx)^{-1}dx = \frac{ln|a + bx|}{b} + C}$$

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = e^{\frac{t}{1.785s} + C_1}$$

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = e^{C_1}e^{\frac{t}{1.785s}}$$

Figure 1.34 Solution of the integral

$$\left|\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{t}{1.785s}}$$

An initial velocity of zero can be assumed to find the value of the integration constant

$$\left|\frac{0 + 35.02\frac{m}{s}}{0 - 35.02\frac{m}{s}}\right| = C_2 e^{\frac{0}{1.785s}} \qquad 1 = C_2$$

This can then be simplified, and the absolute value sign eliminated.

$$\frac{v + 35.02\frac{m}{s}}{v - 35.02\frac{m}{s}} = \pm e^{\frac{t}{1.785s}}$$

$$v + 35.02\frac{m}{s} = \pm v e^{\frac{t}{1.785s}} \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}}$$

$$v\left(1 \mp e^{\frac{t}{1.785s}}\right) = \mp 35.02\frac{m}{s} e^{\frac{t}{1.785s}} - 35.02\frac{m}{s}$$

$$v = 35.02\frac{m}{s}\left(\frac{\mp e^{\frac{t}{1.785s}} - 1}{1 \mp e^{\frac{t}{1.785s}}}\right) \qquad 0 = 35.02\frac{m}{s}\left(\frac{\mp 1 - 1}{1 \mp 1}\right) = \left(\frac{1 - 1}{1 + 1}\right) = \frac{0}{2}$$

$$v = 35.02\frac{m}{s}\left(\frac{e^{\frac{t}{1.785s}} - 1}{1 + e^{\frac{t}{1.785s}}}\right)$$

Figure 1.35 Solution of the integral and application of the initial conditions

As evident from the example, non-linear equations are involved and don't utilize routine methods. Typically the numerical methods discussed in the next chapter are preferred.

1.5.2 Non-Linear Equation Terms

If our models include a device that is non-linear and we want to use a linear technique to solve the equation, we will need to linearize the model before we can proceed. A non-linear system can be approximated with a linear equation using the following method.

1. Pick an operating point or range for the component.
2. Find a constant value that relates a change in the input to a change in the output.
3. Develop a linear equation.
4. Use the linear equation for the analysis.

A linearized differential equation can be approximately solved using known techniques as long as the system doesn’t travel too far from the linearized point. The example in Figure 1.36 shows the linearization of a non-linear equation about a given operating point. This equation will be approximately correct as long as the first derivative doesn’t move too far from 100. When this value does, the new velocity can be calculated.

Assume we have the non-linear differential equation below. It can be solved by linearizing the value about the operating point

Given,

$$\dot{y}^2 + 4y = 200 \qquad y(0) = 10$$

We can make the equation linear by replacing the velocity squared term with the velocity times the actual velocity. As long as the system doesn't vary too much from the given velocity the model should be reasonably accurate.

$$\dot{y} = \pm\sqrt{200 - 4y}$$

$$\dot{y}(0) = \pm\sqrt{200 - 4(10)} = \pm 12.65$$

$$12.65\dot{y} + 4y = 20$$

This system may now be solved as a linear differential equation. If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this.

Homogeneous:

$$12.65\dot{y} + 4y = 0$$

$$12.65A + 4 = 0 \qquad A = -0.316$$

$$y_h = Ce^{-0.316t}$$

Particular:

$$y_p = A$$

$$12.65(0) + 4A = 200 \qquad A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.316t} + 50$$

$$10 = Ce^{0} + 50 \qquad C = -40$$

$$y(t) = -40e^{-0.316t} + 50$$

Figure 1.36 Linearizing a differential equation

If the velocity (first derivative of y) changes significantly, then the differential equation should be changed to reflect this. For example we could decide to recalculate the equation value after 0.1s.

$$y(0.1) = -40e^{-0.316(0.1)} + 50 = 11.24$$

$$\frac{d}{dt}y(0.1) = -40(-0.316)e^{-0.316(0.1)} = 12.25 \qquad \text{Note: a small change}$$

$$12.25y' + 4y = 20$$

Now recalculate the solution to the differential equation.

Homogeneous:

$$12.25\dot{y} + 4y = 0$$

$$12.25A + 4 = 0 \qquad A = -0.327$$

$$y_h = Ce^{-0.327t}$$

Particular:

$$y_p = A$$

$$12.25(0) + 4A = 200 \qquad A = 50$$

Initial conditions:

$$y(t) = Ce^{-0.327t} + 50$$

$$11.24 = Ce^{0.1} + 50 \qquad C = -35.070575$$

$$y(t) = -35.07e^{-0.316t} + 50$$

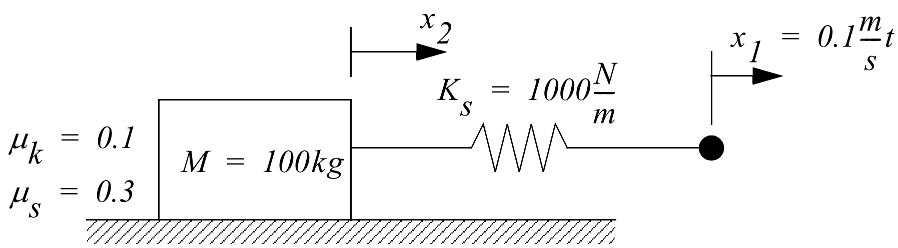
Notice that the values have shifted slightly, and as the analysis progresses the equations will adjust slowly. Higher accuracy can be obtained using smaller steps in time.

Figure 1.37 Linearizing a differential equation

1.5.3 Changing Systems

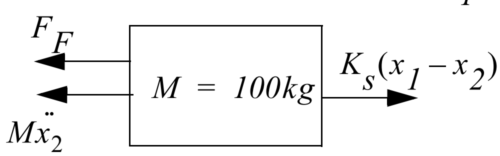
In practical systems, the forces at work are continually changing. For example a system often experiences a static friction force when motion is starting, but once motion starts it is replaced with a smaller kinetic friction. Another example is tension in a cable. When in tension a cable acts as a spring. But, when in compression the force goes to zero.

Consider the example in Figure 1.38. A mass is pulled by a springy cable. The right hand side of the cable is being pulled at a constant rate, while the block is free to move, only restricted by friction forces and inertia. At the beginning all components are at rest and undeflected.



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

For the cable/spring in tension $x_1 - x_2$ Š 0



$$\sum F_x = -F_F + K_s(x_1 - x_2) = M\ddot{x}_2$$

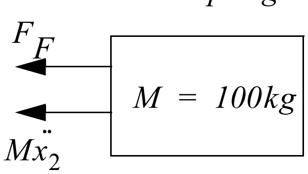
$$-F_F + 1000\frac{N}{m}\left(0.1\frac{m}{s}t - x_2\right) = 100kg\ddot{x}_2$$

$$100kg\ddot{x}_2 + 1000\frac{N}{m}x_2 = 1000\frac{N}{m}0.1\frac{m}{s}t - F_F$$

$$\ddot{x}_2 + 10\frac{N}{kgm}x_2 = 1\frac{N}{kgs}t - \frac{F_F}{100kg}$$

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

For the cable/spring in compression $x_1 - x_2 < 0$



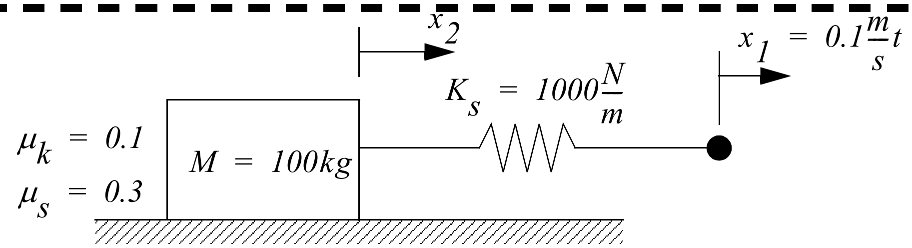
$$\sum F_x = -F_F = M\ddot{x}_2$$

$$-F_F = 100kg\ddot{x}_2$$

$$100kg\ddot{x}_2 = -F_F$$

$$\ddot{x}_2 = -\frac{F_F}{100kg}$$

Figure 1.38 A differential equation for a mass pulled by a springy cable



An FBD and equation can be developed for the system. The friction force will be left as a variable at this point.

$$N = 100kg 9.81\frac{N}{kg} = 981N$$

static friction $\qquad \frac{d}{dt}x_2 \le 0\frac{m}{s}$

$$0N \le F_F < \mu_s N < 294.3N$$

kinetic friction $\qquad \frac{d}{dt}x_2 > 0\frac{m}{s}$

$$F_F = \mu_k N = 98.1N$$

Figure 1.39 Friction forces for the mass

The analysis of the system begins by assuming the system starts at rest and undeflected. In this case the cable/spring will be undeflected with no force, and the mass will be experiencing static friction. Therefore the block will stay in place until the cable stretches enough to overcome the static friction.

$$x_2 = 0 \qquad \ddot{x}_2 = 0 \qquad F_F = 294.3N$$

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{F_F}{100kg}$$

$$0 + 10s^{-2}0 = 1\frac{m}{s^3}t - \frac{294.3N}{100kg}$$

$$1\frac{m}{s^3}t = \frac{294.3kgm}{100kgs^2}$$

$$t = 2.943s$$

Therefore the system is static from 0 to 2.943s

Figure 1.40 Analysis of the object before motion begins

After motion begins the object will only experience kinetic friction, and continue to accelerate until the cable/spring becomes loose in compression. This stage of motion requires the solution of a differential equation.

$$\ddot{x}_2 + 10s^{-2}x_2 = 1\frac{m}{s^3}t - \frac{98.1N}{100kg}$$

For the homogeneous,

$$\ddot{x}_2 + 10s^{-2}x_2 = 0$$

$$A + 10s^{-2} = 0 \qquad A = \pm 3.16js^{-1}$$

$$x_h = C_1 sin(3.16t + C_2)$$

For the particular,

$$x_p = At + B \qquad \dot{x}_p = A \qquad \ddot{x}_p = 0$$

$$0 + 10s^{-2}(At + B) = 1\frac{m}{s^3}t - \frac{98.1N}{100kg}$$

$$10s^{-2}A = 1\frac{m}{s^3} \qquad A = 0.1\frac{m}{s}$$

$$10s^{-2}B = -\frac{98.1N}{100kg} \qquad B = -0.0981m$$

Figure 1.41 Analysis of the object after motion begins

For the initial conditions,

$$x(2.943s) = 0m \qquad \frac{d}{dt}x(2.943s) = 0\frac{m}{s}$$

$$x(t) = C_1 sin(3.16t + C_2) + 0.1\frac{m}{s}t - 0.0981m$$

$$0 = C_1 sin(3.16(2.943s) + C_2) + 0.1\frac{m}{s}(2.943s) - 0.0981m$$

$$C_1 sin(9.29988 + C_2) = -0.1962$$

$$\frac{d}{dt}x(t) = 3.16C_1 cos(3.16t + C_2) + 0.1\frac{m}{s}$$

$$0 = 3.16C_1 cos(3.16(2.943) + C_2) + 0.1\frac{m}{s}$$

$$C_1 cos(9.29988 + C_2) = -0.0316$$

$$\frac{C_1 sin(9.29988 + C_2)}{C_1 cos(9.29988 + C_2)} = \frac{-0.1962}{-0.0316}$$

$$tan(9.29988 + C_2) = 6.209 \qquad C_2 = (-7.889 + \pi n)rad \quad n \text{ Œ } I$$

$$C_1 = \frac{-0.1962}{sin(9.29988 - 7.889)} = -0.199m$$

$$x(t) = -0.199m\,sin(3.16t - 7.889rad) + 0.1\frac{m}{s}t - 0.0981m$$

$$\frac{d}{dt}x(t) = -0.199(3.16)m\,cos(3.16t - 7.889rad) + 0.1\frac{m}{s}$$

Figure 1.42 Analysis of the object after motion begins

The equation of motion changes after the cable becomes slack. This point in time can be determined when the displacement of the block equals the displacement of the cable/spring end.

$$0.1\frac{m}{s}t = -0.199m\sin(3.16t - 7.889rad) + 0.1\frac{m}{s}t - 0.0981m$$

$$-0.199m\sin(3.16t - 7.889rad) = 0.0981m$$

$$3.16t - 7.889 + \pi n = -0.51549413 \qquad t = 3.328s$$

$$x(3.328) = 0.137m \qquad \frac{d}{dt}x(2.333) = 0.648\frac{m}{s}$$

After this the differential equation without the cable/spring is used.

$$x_2'' = -\frac{98.1N}{100kg} = -0.981\frac{m}{s^2}$$

$$\dot{x}_2 = \left(-0.981\frac{m}{s^2}\right)t + C_1$$

$$0.648\frac{m}{s} = \left(-0.981\frac{m}{s^2}\right)(3.328s) + C_1$$

$$C_1 = 3.913\frac{m}{s}$$

$$x_2 = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)t^2 + 3.913\frac{m}{s}t + C_2$$

$$0.137m = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)(3.328s)^2 + 3.913\frac{m}{s}(3.328s) + C_2$$

$$C_2 = -7.453m$$

$$x_2(t) = \left(-\frac{0.981}{2}\frac{m}{s^2}\right)t^2 + 3.913\frac{m}{s}t - 7.453m$$

This motion continues until the block stops moving.

$$0 = \left(-0.981\frac{m}{s^2}\right)t + 3.913\frac{m}{s}$$

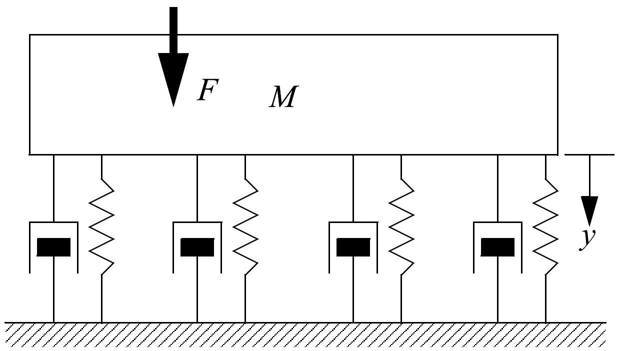
$$t = 3.989s$$

The solution can continue, considering when to switch the analysis conditions.

Figure 1.43 Determining when the cable become slack

1.6 Case Study

A typical vibration control system design is described in Figure 1.44.



The model to the left describes a piece of reciprocating industrial equipment. The mass of the equipment is 10000kg. The equipment operates such that a force of 1000N with a frequency of 2Hz is exerted on the mass. We have been asked to design a vibration isolation mounting system. The criteria we are given is that the mounts should be 30cm high when unloaded, and 25cm when loaded with the mass. In addition, the oscillations while the machine is running cannot be more than 2cm total. In total there will be four mounts mounted around the machine. Each isolator will be composed of a spring and a damper.

Figure 1.44 A vibration control system

There are a number of elements to the design and analysis of this system, but as usual the best place to begin is by developing a free body diagram, and a differential equation. This is done in Figure 1.45.

$$+\downarrow \sum F_y = F - 4K_s y - 4K_d \dot{y} + Mg = M\ddot{y}$$

$$M\ddot{y}' + 4K_d\dot{y} + 4K_s y = F + Mg$$

$$\ddot{y} + \frac{4K_d\dot{y}}{M} + \frac{4K_s y}{M} = \frac{F}{M} + g$$

$$\ddot{y} + \frac{4K_d\dot{y}}{10000Kg} + \frac{4K_s y}{10000Kg} = \frac{1000N}{10000Kg}sin(2(2\pi)t) + 9.81ms^{-2}$$

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 0.0004Kg^{-1}K_s y = 0.1ms^{-2}sin(4\pi t) + 9.81ms^{-2}$$

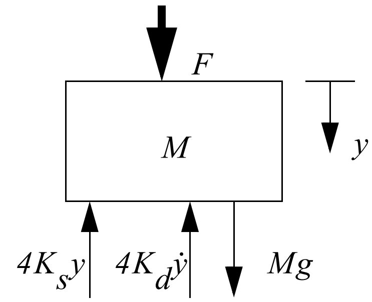


Figure 1.45 FBD and derivation of equation

Using the differential equation, the spring values can be found by assuming the machine is at rest. This is done in Figure 1.46.

When the system is at rest the equation is simplified; the acceleration and velocity terms both become zero. In addition, we will assume that the cyclic force is not applied for the unloaded/loaded case. This simplifies the differential equation by eliminating several terms.

$$0.0004Kg^{-1}K_s y = 9.81ms^{-2}$$

Now we can consider that when unloaded the spring is 0.30m long, and after loading the spring is 0.25m long. This will result in a downward compression of 0.05m, in the positive y direction.

$$0.0004Kg^{-1}K_s(0.05m) = 9.81ms^{-2}$$

$$K_s = \frac{9.81}{0.0004(0.05)}Kgms^{-2}m^{-1}$$

$$\therefore K_s = 491KNm^{-1}$$

Figure 1.46 Calculation of the spring coefficient

The remaining unknown is the damping coefficient. At this point we have determined the range of motion of the mass. This can be done by developing the particular solution of the differential equation, as it will contain the steady-state oscillations caused by the forces as shown in Figure 1.47.

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 0.0004Kg^{-1}(491KNm^{-1})y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

$$\ddot{y} + 0.0004Kg^{-1}K_d\dot{y} + 196s^{-2}y = 0.1ms^{-2}\sin(4\pi t) + 9.81ms^{-2}$$

The particular solution can now be found by guessing a value, and solving for the coefficients. (Note: The units in the expression are uniform (i.e., the same in each term) and will be omitted for brevity.)

$$y = A\sin(4\pi t) + B\cos(4\pi t) + C$$

$$y' = 4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t)$$

$$y'' = -16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)$$

$$\therefore (-16\pi^2 A\sin(4\pi t) - 16\pi^2 B\cos(4\pi t)) + 0.0004K_d(4\pi A\cos(4\pi t) - 4\pi B\sin(4\pi t))$$
$$+ 196(A\sin(4\pi t) + B\cos(4\pi t) + C) = 0.1\sin(4\pi t) + 9.81$$

$$-16\pi^2 B + 0.0004K_d 4\pi A + 196A = 0$$

$$B = A(31.8 \times 10^{-6}K_d + 1.24)$$

$$-16\pi^2 A + 0.0004K_d(-4\pi B) + 196A = 0.1$$

$$A(-16\pi^2 + 196) + B(-5.0 \times 10^{-3}K_d) = 0.1$$

$$A(-16\pi^2 + 196) + A(31.8 \times 10^{-6}K_d + 1.24)(-5.0 \times 10^{-3}K_d) = 0.1$$

$$A = \frac{0.1}{-16\pi^2 + 196 + (31.8 \times 10^{-6}K_d + 1.24)(-5.0 \times 10^{-3}K_d)}$$

$$A = \frac{0.1}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

$$B = \frac{3.18 \times 10^{-6}K_d - 0.124}{K_d^2(-159 \times 10^{-9}) + K_d(-6.2 \times 10^{-3}) + 38.1}$$

$$C = 9.81ms^{-2}$$

Figure 1.47 Particular solution of the differential equation

The particular solution can be used to find a damping coefficient that will give an overall oscillation of 0.02m, as shown in Figure 1.48. In this case Mathcad was used to find the solution, although it could have also been found by factoring out the algebra, and finding the roots of the resulting polynomial.

In the previous particular solution the values were split into cosine and sine components. The magnitude of oscillation can be calculated with the Pythagorean formula.

$$magnitude = \sqrt{A^2 + B^2}$$

$$magnitude = \frac{\sqrt{(0.1)^2 + \left((3.18 \cdot 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \cdot 10^{-9}) + K_d(-(6.2 \cdot 10^{-3})) + 38.1}$$

The design requirements call for a maximum oscillation of 0.02m, or a magnitude of 0.01m.

$$0.01 = \frac{\sqrt{(0.1)^2 + \left((3.18 \cdot 10^{-6})K_d - 0.124\right)^2}}{K_d^2(-159 \cdot 10^{-9}) + K_d(-(6.2 \cdot 10^{-3})) + 38.1}$$

A given-find block was used in Mathcad to obtain a damper value of,

$$K_d = 3411N\frac{s}{m}$$

Aside: the Mathcad solution

$$f(k) := \frac{\sqrt{0.01 + \left[\left(3.18 \cdot 10^{-6} \cdot k\right) - 0.124\right]^2}}{\left[k \cdot k \cdot \left(-159 \cdot 10^{-9}\right)\right] + k \cdot \left(-6.2 \cdot 10^{-3}\right) + 38.1}$$

$$k_d := 1$$

given

$$f(k_d) = 0.01$$

$$\text{find}(k_d) = 3.411 \times 10^3$$

Figure 1.48 Determining the damping coefficient

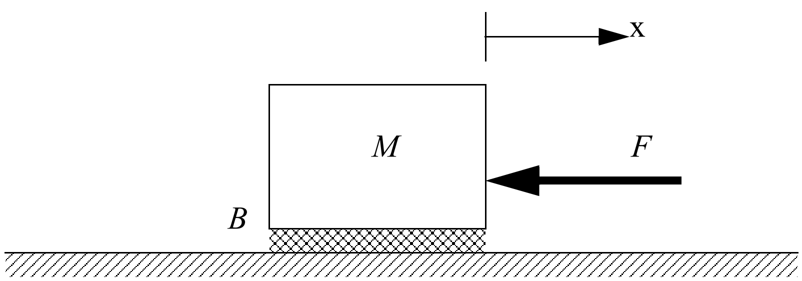
The values of the spring and damping coefficients can be used to select actual components. Some companies will design and build their own components. Components can also be acquired by searching catalogs, or requesting custom designs from other companies.

1.7 Summary

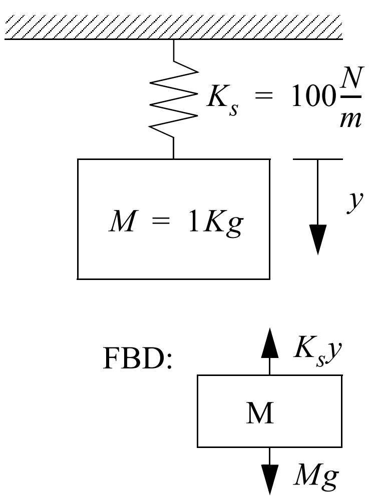
- First and second-order differential equations were analyzed explicitly.
- First and second-order responses were examined.
- The topic of analysis was discussed.
- A case study looked at a second-order system.
- Non-linear systems can be analyzed by making them linear.

1.8 Problems

1. The mass, M, illustrated below starts at rest. It can slide across a surface, but the motion is opposed by viscous friction (damping) with the coefficient B. Initially the system starts at rest, when a constant force, F, is applied. Write the differential equation for the mass, and solve the differential equation. Leave the results in variable form.



2. The following differential equation was derived for a mass suspended with a spring. At time 0s the system is released and allowed to drop. It then oscillates. Solve the differential equation to find the motion as a function of time.



$$+\uparrow \quad \sum F_y = K_s y - Mg = -M\ddot{y}$$

$$\left(100\frac{N}{m}\right)y - (1Kg)\left(9.81\frac{N}{Kg}\right) = (-1Kg)\ddot{y}$$

$$\left(1\frac{Nm}{s^2}\right)\ddot{y} + \left(100\frac{N}{m}\right)y = 9.81N$$

$$(1Kg)\ddot{y} + \left(100\frac{Kgm}{ms^2}\right)y = 9.81\frac{Kgm}{s^2}$$

$$\ddot{y} + (100s^{-2})y = 9.81ms^{-2}$$

$$y_0 = 0m \qquad \dot{y}_0 = 0ms^{-1}$$

3. Solve the following differential equation with the three given cases. All of the systems have a

step input 'y' and start undeflected and at rest.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = y$$

initial conditions $\dot{x} = 0$, $x = 0$, $y = 1$

case 1: $\xi = 0.5$ $\omega_n = 10$

case 2: $\xi = 1$ $\omega_n = 10$

case 3: $\xi = 2$ $\omega_n = 10$

4. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2$$

initial conditions $V_i = 5$, $V_o = 0$, $\dot{V}_o = 0$

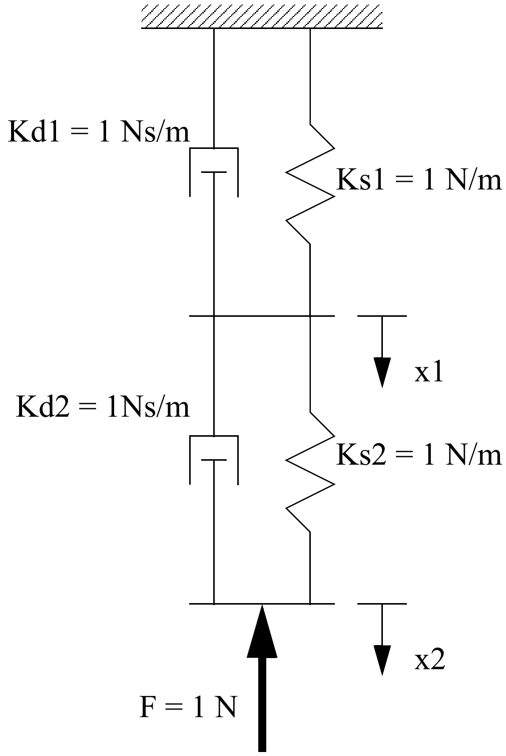
5. Solve the following differential equation with the given initial conditions and draw a sketch of the first 5 seconds. The input is a step function that turns on at t=0.

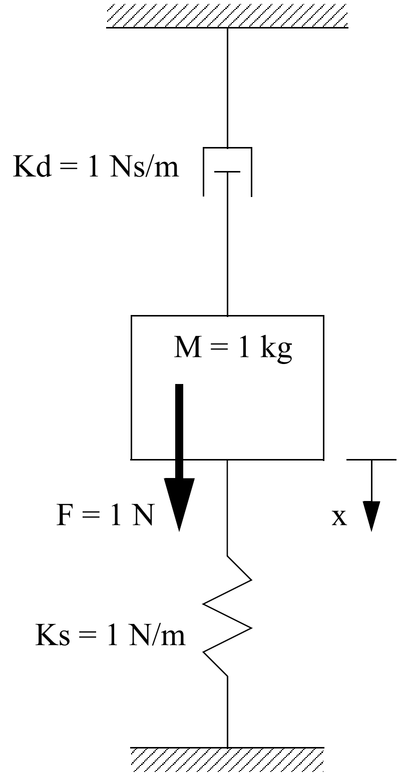
$$0.5\ddot{V}_o + 0.6\dot{V}_o + 2.1V_o = 3V_i + 2$$

initial conditions $V_i = 5$, $V_o = 0$, $\dot{V}_o = 1$

6. a) Write the differential equations for the system below. Solve the equations for x assuming that

the system is at rest and undeflected before t=0. Also assume that gravity is present.





b) State whether each system is first or second-order. If the system if first-order find the time constant. If it is second-order find the natural frequency and damping ratio.

7. Solve the following differential equation with the three given cases. All of the systems have a sinusoidal input 'y' and start undeflected and at rest.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = y$$

initial conditions $\dot{x} = 0$, $x = 0$, $y = \sin(t)$

case 1: $\xi = 0.5$ $\omega_n = 10$

case 2: $\xi = 1$ $\omega_n = 10$

case 3: $\xi = 2$ $\omega_n = 10$

8. A spring damper system supports a mass of 34N. If it has a spring constant of 20.6N/cm, what is the systems natural frequency?

9. Using a standard lumped parameter model the weight is 36N, stiffness is $2.06*10^3$ N/m and damping is 100Ns/m. What are the natural frequency (Hz) and damping ratio?

10. What is the differential equation for a second-order system that responds to a step input with an overshoot of 20%, with a delay of 0.4 seconds to the first peak?

11. A system is to be approximated with a mass-spring-damper model using the following parameters: weight 28N, viscous damping 6Ns/m, and stiffness 36N/m. Calculate the undamped natural frequency (Hz) of the system, the damping ratio and describe the type of response you would expect if the mass were displaced and released. What additional damping would be required to make the system critically damped?

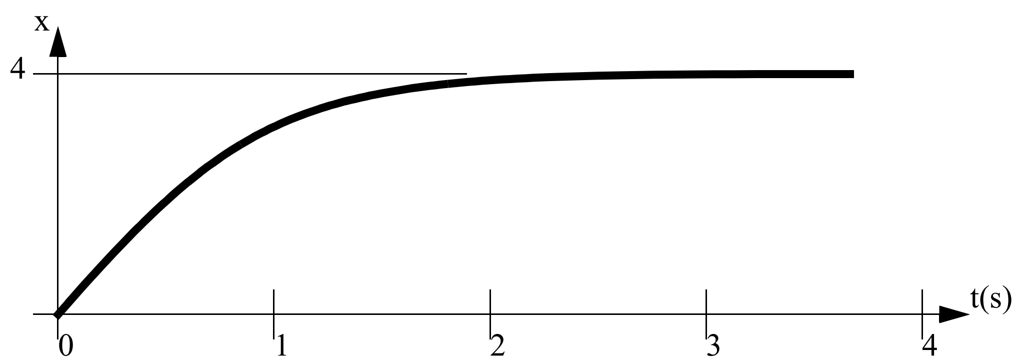
$$M\ddot{x} + K_d\dot{x} + K_s x = F$$

12. Solve the differential equation below using homogeneous and particular solutions. Assume the system starts undeflected and at rest.

$$\dddot{\theta} + 40\ddot{\theta} + 20\dot{\theta} + 2\theta = 4$$

13. What would the displacement amplitude after 100ms for a system having a natural frequency of 13 rads/sec and a damping ratio of 0.20. Assume an initial displacement of 50mm, and a steady state displacement of 0mm. (Hint: Find the response as a function of time.)

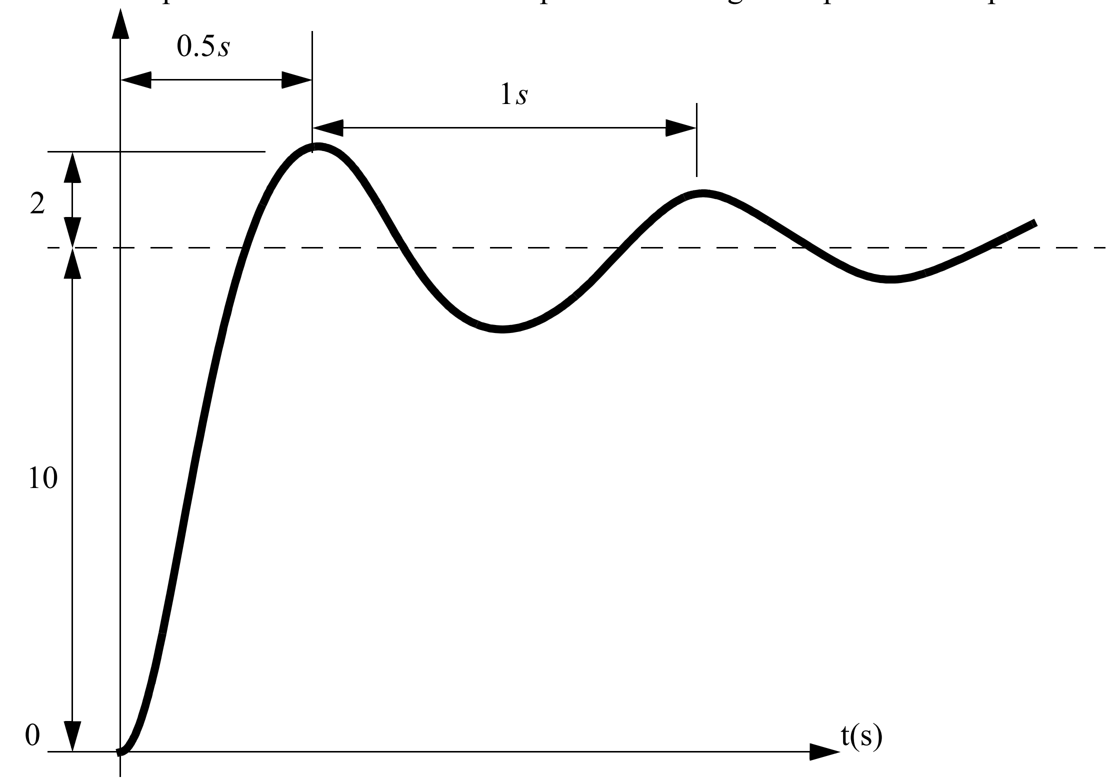
14. Determine the first order differential equation given the graphical response shown below. Assume the input is a step function.



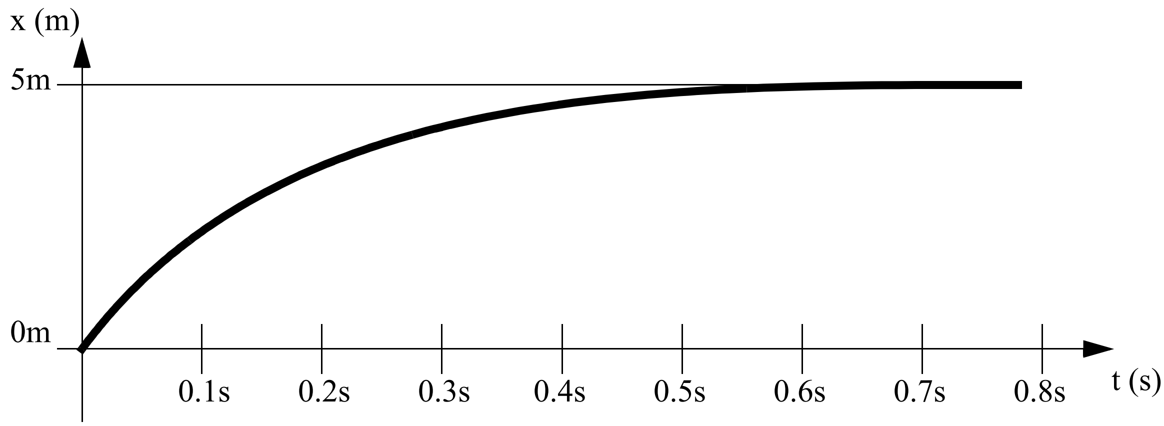
15. Explain with graphs how to develop first and second-order equations using experimental data.

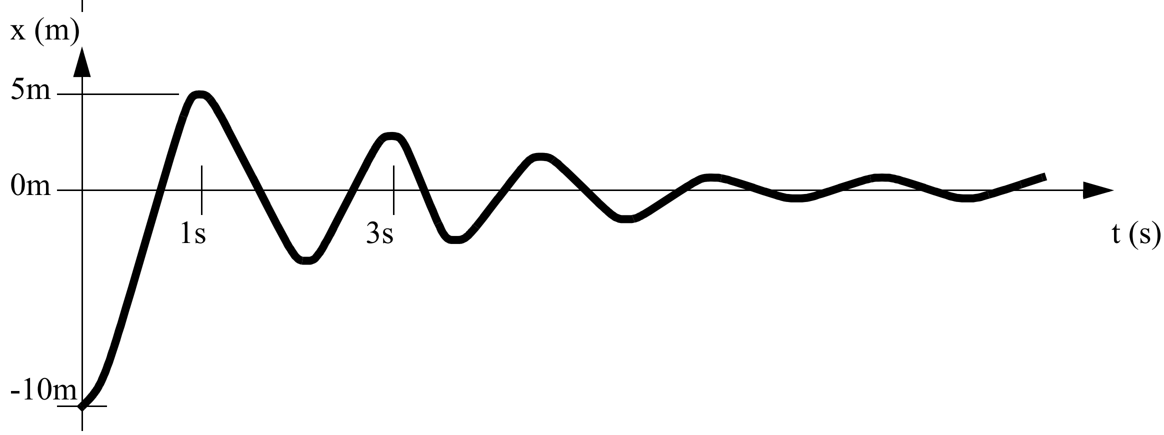
16. The second order response below was obtained experimentally. Determine the parameters of

the differential equation that resulted in the response assuming the input was a step function.



17. Develop equations (function of time) for the first and second order responses shown below.





18. A mass-spring-damper system has a mass of 10Kg and a spring coefficient of 1KN/m. Select a damping coefficient so that the system will have an overshoot of 20% for a step input.

1.9 Problems Solutions

1.

$$x(t) = \frac{-FM}{B^2} e^{-\frac{B}{M}t} - \frac{F}{B}t + \frac{FM}{B^2}$$

2.

homogeneous: guess $y_h = e^{At}$ $\qquad \dot{y}_h = Ae^{At}$ $\qquad \ddot{y}_h = A^2e^{At}$

$$A^2e^{At} + (100s^{-2})e^{At} = 0$$

$$A^2 = -100s^{-2} \qquad A = \pm 10js^{-1}$$

$$y_h = C_1\cos(10t + C_2)$$

particular: guess $y_p = A$ $\qquad \dot{y}_p = 0$ $\qquad \ddot{y}_p = 0$

$$(0) + (100s^{-2})A = 9.81ms^{-2}$$

$$(100s^{-2})A = 9.81ms^{-2}$$

$$A = \frac{9.81ms^{-2}}{100s^{-2}} = 0.0981m$$

$$y_p = 0.0981m$$

initial conditions:

$$y = y_h + y_p = C_1\cos(10t + C_2) + 0.0981m$$

$$y' = -10C_1\sin(10t + C_2)$$

for d/dt y0 = 0m:

$$0 = -10C_1\sin(10(0) + C_2) \qquad C_2 = 0$$

for y0 = 0m:

$$0 = C_1\cos(10(0) + (0)) + 0.0981m$$

$$-0.0981m = C_1\cos(0) \qquad C_1 = -0.0981m$$

$$\boxed{y(t) = (-0.0981m)\cos(10t) + 0.0981m}$$

3.

case 1: $x(t) = -0.0115e^{-5t}\cos(8.66t - 0.524) + 0.010$

case 2: $x(t) = -0.010e^{-10t} - 0.10te^{-10t} + 0.010$

case 3: $x(t) = 775 \cdot 10^{-6}e^{-37.32t} - 0.0108e^{-2.679t} + 0.010$

4.

$$V_o(t) = -8.465e^{-0.6t}\sin(1.960t + 1.274) + 8.095$$

or

$$V_o(t) = -8.465e^{-0.6t}\cos(1.960t - 0.2971) + 8.095$$

5.

$$V_0(t) = -8.331e^{-0.6t}\cos(1.96t - 0.238) + 8.095$$

6.

a)

$$x_1(t) = e^{-t} - 1$$

$$x_2(t) = 2e^{-t} - 2$$

$$\tau = 1$$

b)

$$x(t) = -12.485e^{-0.5t}\cos(0.866t - 0.524) + 10.81$$

$$\zeta = 0.5 \qquad \omega_n = 1$$

7.

case 1: $x(t) = -0.00117e^{-5t}\sin(8.66t - 1.061) + 0.0101\sin(t - 0.101)$

case 2: $x(t) = (1.96 \cdot 10^{-3})e^{-10t} + (9.9 \cdot 10^{-3})te^{-10t} + (9.9 \cdot 10^{-3})\sin(t + 0.20)$

case 3: $x(t) = (3.5 \cdot 10^{-3})e^{-2.679t} - (18 \cdot 10^{-6})e^{-37.32t} + (9.4 \cdot 10^{-3})\sin(t + 0.382)$

8. 24.37 rad/sec

9. fn=3.77Hz, damp.=.575

10.

$$\ddot{x} + 8.048\dot{x} + 77.88x = F(t)$$

11.

Given $K_d = 6N\frac{s}{m}$ $K_s = 36\frac{N}{m}$ $M = \frac{28N}{9.81\frac{N}{kg}} = 2.85kg$

The typical transfer function for a mass-spring-damper systems is,

$$\ddot{x} + \dot{x}\left(\frac{K_d}{M}\right) + x\left(\frac{K_s}{M}\right) = \frac{F}{M}$$

The second order parameters can be calculated from this.

$$\ddot{x} + \dot{x}(2\zeta\omega_n) + x(\omega_n^2) = y(t)$$

$$\omega_n = \sqrt{\frac{K_s}{M}} = \sqrt{\frac{36\frac{N}{m}}{2.85kg}} = \sqrt{\frac{36\frac{kgm}{ms^2}}{2.85kg}} = \sqrt{12.63s^{-2}} = 3.55\frac{rad}{s} = 0.6Hz$$

$$\xi = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{6N\frac{s}{m}}{2(3.55)\frac{rad}{s}2.85kg} = 0.296$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.39\frac{rad}{s}$$

If pulled and released the system would have a decaying oscillation about 0.52Hz

A critically damped system would require a damping coefficient of....

$$\xi = \frac{\left(\frac{K_d}{M}\right)}{2\omega_n} = \frac{K_d}{2(3.55)\frac{rad}{s}2.85kg} = 1.00 \qquad K_d = 20.2\frac{Ns}{m}$$

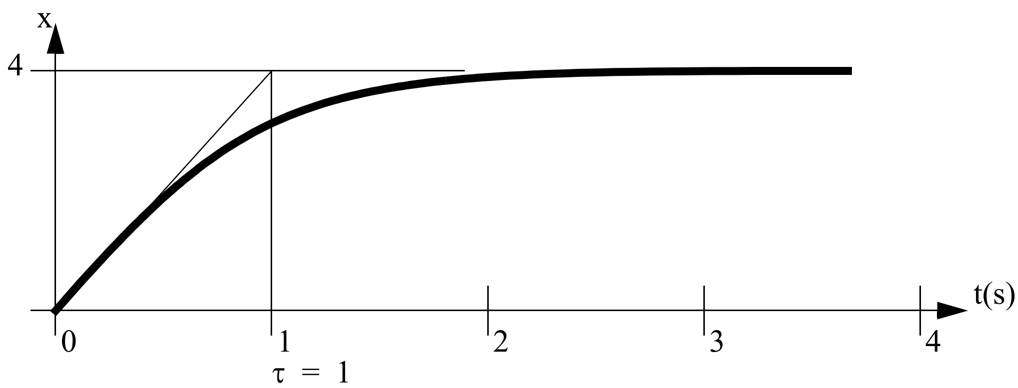
12.

$$\theta(t) = -66 \cdot 10^{-6}e^{-39.50t} - 3.216e^{0.1383t} + 1.216e^{-0.3368t} + 2.00$$

13.

$$y(t) = 0.05e^{-2.6t}\cos(12.74t) = 0.05e^{-2.6t}\sin\left(12.74t + \frac{\pi}{2}\right)$$

14.



Given the equation form,

$$\dot{x} + \frac{1}{\tau}x = A$$

The values at steady state will be

$$\dot{x} = 0 \qquad x = 4$$

So the unknown 'A' can be calculated.

$$0 + \frac{1}{1}4 = A \qquad A = 4$$

$$\dot{x} + \frac{1}{1}x = 4$$

$$\dot{x} + x = 4$$

15.

Key points:

First-order:
find initial final values
find time constant with 63% or by slope
use these in standard equation

Second-order:
find damped frequency from graph
find time to first peak
use these in cosine equation

16.

For the first peak:

$$\frac{b}{\Delta x} = e^{-\sigma t_p}$$

$$\frac{2}{10} = e^{-\sigma 0.5} \qquad\qquad \omega_d \approx \frac{\pi}{t_p}$$

$$\ln\left(\frac{2}{10}\right) = -\sigma 0.5$$

$$\sigma = -2\ln\left(\frac{2}{10}\right) = 3.219$$

$$\xi = \frac{1}{\sqrt{\left(\frac{\pi}{t_p\sigma}\right)^2 + 1}}$$

For the damped frequency:

$$\omega_d = \frac{2\pi}{1s} = 2\pi$$

These values can be used to find the damping coefficient and natural frequency

$$\sigma = \xi\omega_n \qquad \omega_n = \frac{3.219}{\xi}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$2\pi = \frac{3.219}{\xi}\sqrt{1-\xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 = \frac{1-\xi^2}{\xi^2}$$

$$\left(\frac{2\pi}{3.219}\right)^2 + 1 = \frac{1}{\xi^2} \qquad\qquad \xi = \sqrt{\frac{1}{\left(\frac{2\pi}{3.219}\right)^2 + 1}} = 0.4560$$

$$\omega_n = \frac{3.219}{\xi} = \frac{3.219}{0.4560} = 7.059$$

This leads to the final equation using the steady state value of 10

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = F$$

$$\ddot{x} + 2(0.4560)(7.059)\dot{x} + (7.059)^2 x = F$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = F$$

$$(0) + 6.438(0) + 49.83(10) = F \qquad\qquad F = 498.3$$

$$\ddot{x} + 6.438\dot{x} + 49.83x = 498.3$$

17.

$$x(t) = 5\left(1 - e^{\frac{-t}{0.18}}\right) \qquad\qquad x(t) = -10e^{-0.693t}\cos(\pi t)$$

1.10 Challenge Problems

1. Write a Scilab program to solve first and second order differential equations for step inputs. The program should accept coefficients for the differential equations and initial conditions. It should then produce a function of time.

2. ELECTRICAL AND COMPUTER ENGINEERING REVIEW

Topics:

-

Objectives:

-

2.1 Introductions

2.2 Examples

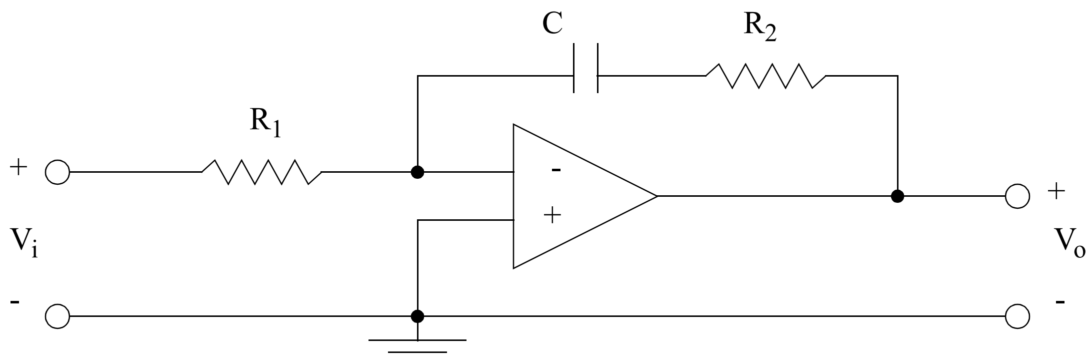
2.3 Summary

2.4 References/Bibliography

2.5 Problems

F.E. Chapter 8 - Differential Equations - Problems , 6-11, 14
F.E. Chapter 39 - Complex Numbers and Electrostatics - Problems 1-5
F.E. Chapter 40 - Direct Current Circuits - Problems 1-32
F.E. Chapter 41 - Alternating Current Circuits - Problems 1-17
F.E. Chapter 42 - Three-Phase Systems and Electronics - Problems 1-13
F.E. Chapter 43 - Computer Hardware - Problems 1-17
F.E. Chapter 44 - Computer Software - Problems 1-32
F.E. Chapter 46 - Controls - Problems 1-5

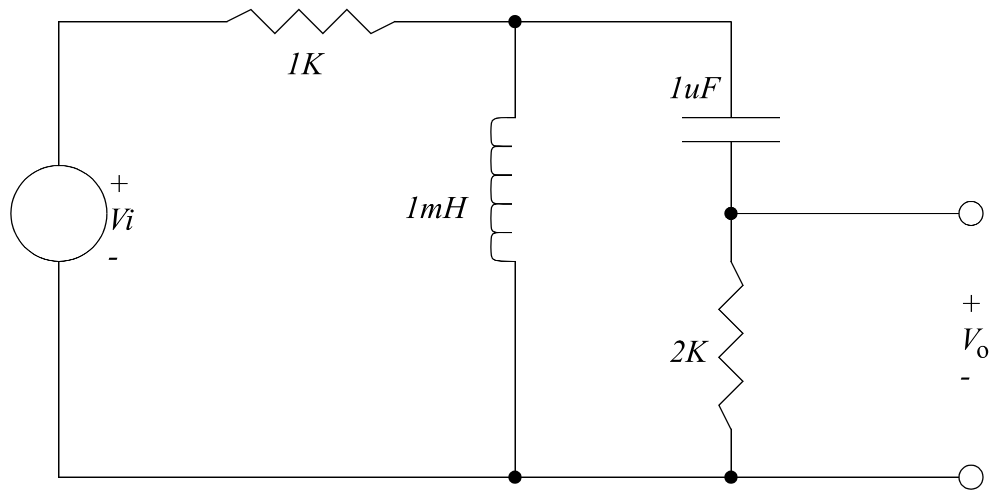
1. Use the loop or node voltage method to write equations for the circuit shown. Write a Scilab program to solve the circuit as a function of frequency.



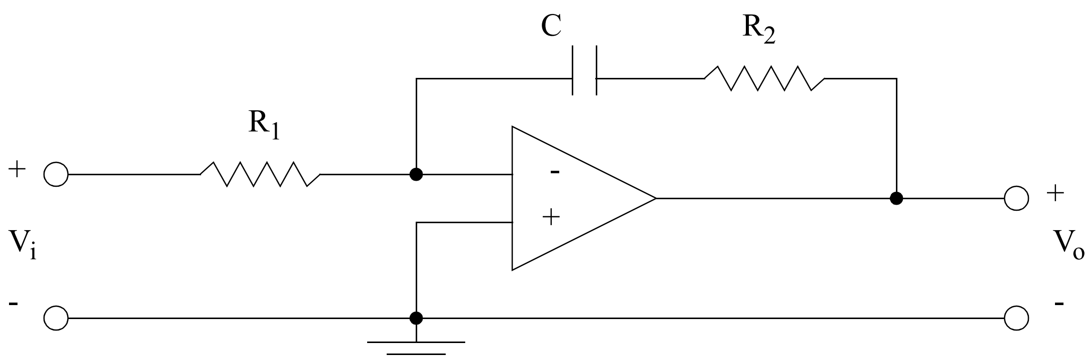
2.6 Challenge Problems

1. Write a program that would allow the specification of a transfer function using points on a Bode plot. The transfer function order should be adjusted to allow the number of points specified.

2. Calculate the output Voltage as a function of the input voltage using Matrices in Scilab.



3. Calculate the output Voltage as a function of the input voltage using State Equations and numerical methods in Scilab.



3. MECHANICAL ENGINEERING REVIEW

Topics:

-

Objectives:

-

3.1 Introduction

3.2 Examples

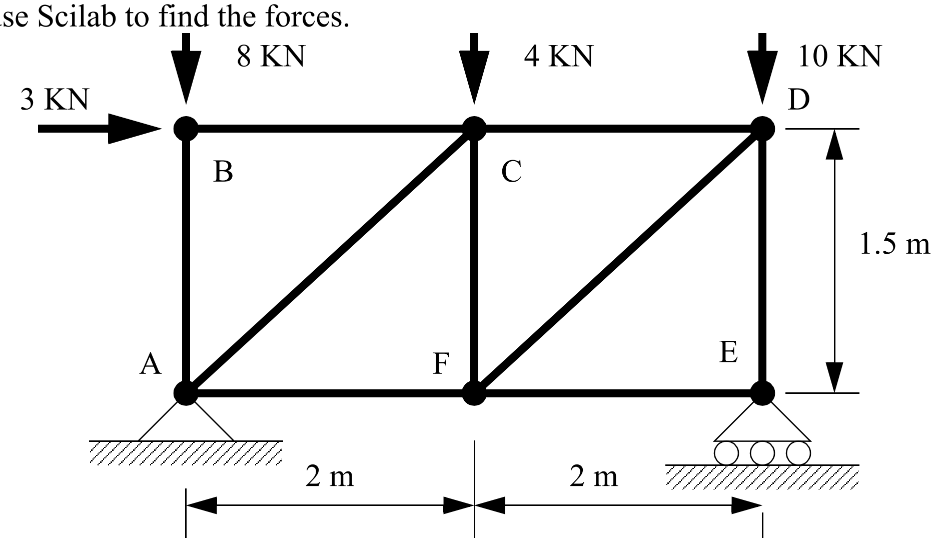
3.3 Summary

3.4 References/Bibliography

3.5 Problems

F.E. Chapter 10 - Systems of Forces - Problems 1-23
F.E. Chapter 11 - Trusses - Problems 1-10
F.E. Chapter 12 - Pulleys, Cables, and Friction - Problems 1-12
F.E. Chapter 13 - Centroids and Momemnts of Inertia - Problems 1-18
F.E. Chapter 18 - Stress and Strain - Problems 1-18
F.E. Chapter 19 - Thermal, Hoop and Torsional Stress - Problems 1-18
F.E. Chapter 20 - Beams - Problems 1-15
F.E. Chapter 21 - Columns - Problems 1-10
F.E. Chapter 36 - Crystallography and Atomic Bonding - Problems 1-8
F.E. Chapter 37 - Material Testing - Problems 1-11
F.E. Chapter 38 - Metallurgy - Problems 1-18

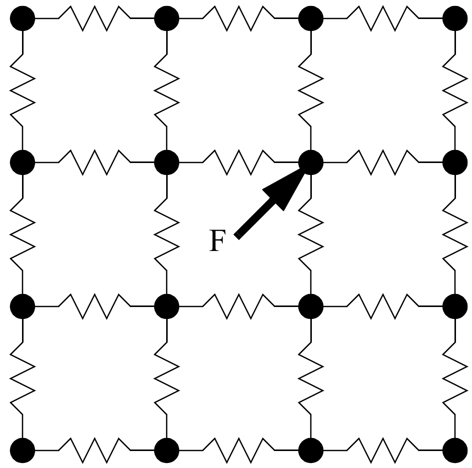
1. Write equationd for the following truss using the method of joints. Enter the equations in a matrix and use Scilab to find the forces.



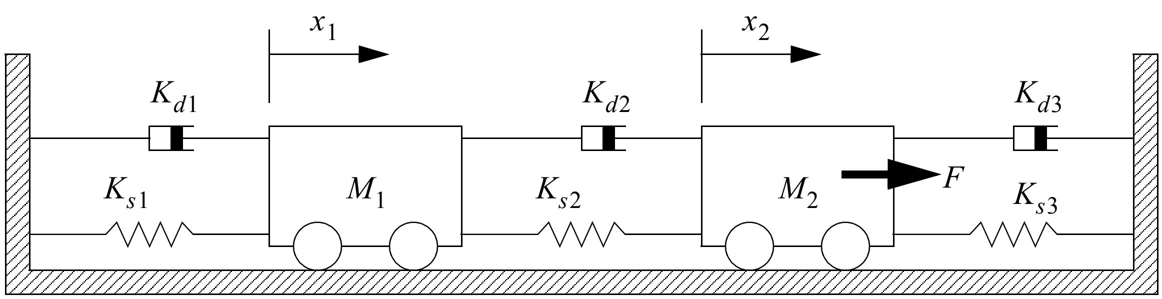
3.6 Challenege Problems

1. Write a program to find the deformation of an array of springs (as pictured below) with a force applied to an internal node. Use a matrix method to solve the problem. Assume that the

nodes at the four corners are fixed and unable to move.



2. Write the differential equations for one of the following systems. Simulate the system response using a numerical method, such as Runge-Kutta integration.



4. INDUSTRIAL ENGINEERING OVERVIEW

Topics:

-

Objectives:

-

4.1 Introduction

4.2 Examples

4.3 Summary

4.4 References/Bibliography

4.5 Problems

F.E. Chapter 10 - Systems of Forces - Problems 1-23
F.E. Chapter 11 - Trusses - Problems 1-10
F.E. Chapter 12 - Pulleys, Cables, and Friction - Problems 1-12
F.E. Chapter 13 - Centroids and Momemnts of Inertia - Problems 1-18
F.E. Chapter 18 - Stress and Strain - Problems 1-18
F.E. Chapter 19 - Thermal, Hoop and Torsional Stress - Problems 1-18
F.E. Chapter 20 - Beams - Problems 1-15
F.E. Chapter 21 - Columns - Problems 1-10
F.E. Chapter 36 - Crystallography and Atomic Bonding - Problems 1-8
F.E. Chapter 37 - Material Testing - Problems 1-11
F.E. Chapter 38 - Metallurgy - Problems 1-18

1. The data set below was obtained over a two week period for a 1.000” shaft with a tolerance of +/- 0.010”. Write a program to automatically update the X-bar, UCL/LCL values given new values. When a new set of values is entered the program should check to see if the process is in control. .

Date	Samples			
Nov., 1, 1994	1.0034”	0.9999”	0.9923”	1.0093”
Nov., 2, 1994	0.9997”	1.0025”	0.9993”	0.9938”
Nov., 3, 1994	1.0001”	1.0009”	0.9997”	1.0079”
Nov., 4, 1994	1.0064”	0.9934”	1.0034”	1.0064”
Nov., 5, 1994	0.9982”	0.9987”	0.9990”	0.9957”
Nov., 6, 1994	0.9946”	1.0101”	1.0000”	0.9974”
Nov., 7, 1994	1.0033”	1.0011”	1.0031”	0.9935”
Nov., 8, 1994	1.0086”	0.9945”	1.0045”	1.0034”
Nov., 9, 1994	0.9997”	0.9969”	1.0067”	0.9972”
Nov., 10, 1994	0.9912”	1.0011”	0.9998”	0.9986”
Nov., 11, 1994	1.0013”	1.0031”	0.9992”	1.0054”
Nov., 12, 1994	1.0027”	1.0000”	0.9976”	1.0038”
Nov., 13, 1994	1.0002”	1.0002”	0.9943”	1.0001”
Nov., 14, 1994	0.9956”	1.0001”	0.9965”	0.9973”

5. PRODUCT DESIGN AND MANUFACTURING REVIEW

Topics:

-

Objectives:

-

5.1 Introduction

5.2 Examples

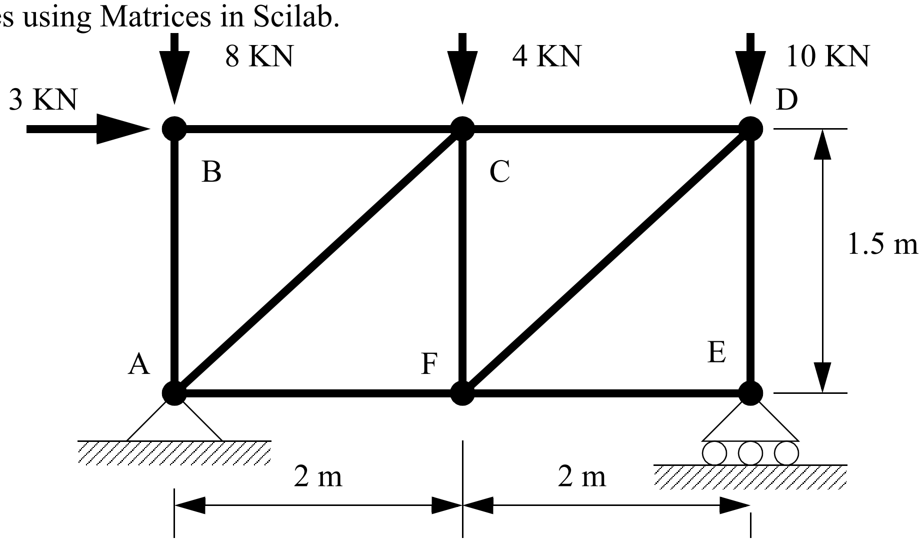
5.3 Summary

5.4 References/Bibliography

5.5 Problems

F.E. Chapter 10 - Systems of Forces - Problems 1-23
F.E. Chapter 11 - Trusses - Problems 1-10
F.E. Chapter 12 - Pulleys, Cables, and Friction - Problems 1-12
F.E. Chapter 13 - Centroids and Momemnts of Inertia - Problems 1-18
F.E. Chapter 18 - Stress and Strain - Problems 1-18
F.E. Chapter 19 - Thermal, Hoop and Torsional Stress - Problems 1-18
F.E. Chapter 20 - Beams - Problems 1-15
F.E. Chapter 21 - Columns - Problems 1-10
F.E. Chapter 36 - Crystallography and Atomic Bonding - Problems 1-8
F.E. Chapter 37 - Material Testing - Problems 1-11
F.E. Chapter 38 - Metallurgy - Problems 1-18

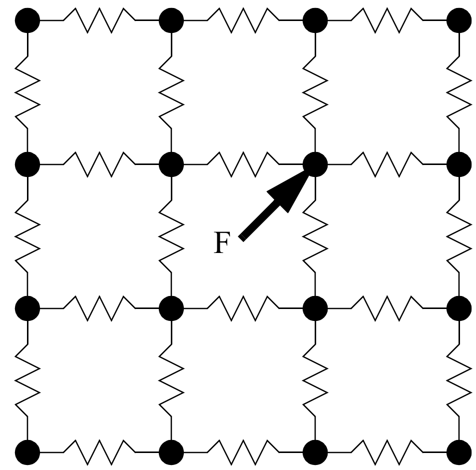
1. Find the forces using Matrices in Scilab.



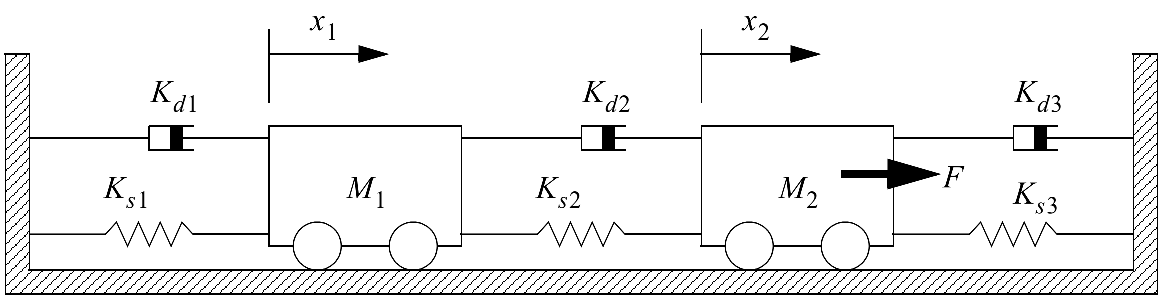
5.6 Challenege Problems

1. Write a program to find the deformation of an array of springs (as pictured below) with a force applied to an internal node. Use a matrix method to solve the problem. Assume that the

nodes at the four corners are fixed and unable to move.



2. Write the differential equations for one of the following systems. Simulate the system response using a numerical method, such as Runge-Kutta integration.



2. BOOLEAN LOGIC DESIGN

Topics:
- Boolean algebra
- Converting between Boolean algebra and logic gates
- Logic examples

Objectives:
- Be able to simplify designs with Boolean algebra and Karnaugh maps

2.1 Introduction

Boolean algebra provides the tools needed to analyze and design logical systems.

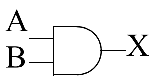
2.2 Boolean Algebra

Boolean algebra was developed in the 1800's by James Bool, an Irish mathematician. It was found to be extremely useful for designing digital circuits, and it is still heavily used by electrical engineers and computer scientists. The techniques can model a logical system with a single equation. The equation can then be simplified and/or manipulated into new forms. The same techniques developed for circuit designers adapt very well to circuit and program.

Boolean equations consist of variables and operations and look very similar to normal algebraic equations. The three basic operators are AND, OR and NOT; more complex operators include exclusive or (EOR), not and (NAND), not or (NOR). Small truth tables for these functions are shown in Figure 2.1. Each operator is shown in a simple equation with the variables A and B being used to calculate a value for X. Truth tables are a simple (but bulky) method for showing all of the possible combinations that will turn an output on or off.

Note: By convention a false state is also called off or 0 (zero). A true state is also called on or 1.

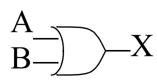
AND



$X = A \cdot B$

A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

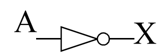
OR



$X = A + B$

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

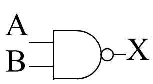
NOT



$X = \overline{A}$

A	X
0	1
1	0

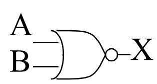
NAND



$X = \overline{A \cdot B}$

A	B	X
0	0	1
0	1	1
1	0	1
1	1	0

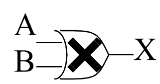
NOR



$X = \overline{A + B}$

A	B	X
0	0	1
0	1	0
1	0	0
1	1	0

EOR



$X = A \oplus B$

A	B	X
0	0	0
0	1	1
1	0	1
1	1	0

Note: The symbols used in these equations, such as + for OR are not universal standards and some authors will use different notations.

Note: The EOR function is available in gate form, but it is more often converted to its equivalent, as shown below.

$$X = A \oplus B = A \cdot \overline{B} + \overline{A} \cdot B$$

Figure 2.1 Boolean Operations with Truth Tables and Gates

In a Boolean equation the operators will be put in a more complex form as shown in Figure 2.2. The variable for these equations can only have a value of 0 for false, or 1 for true. The solution of the equation follows rules similar to normal algebra. Parts of the equation inside parenthesis are to be solved first. Operations are to be done in the sequence NOT, AND, OR. In the example the NOT function for C is done first, but the NOT over the first set of parentheses must wait until a single value is available. When there is a choice the AND operations are done before the OR operations. For the given set of variable values the result of the calculation is false.

given

$$X = \overline{(A + B \cdot C)} + A \cdot (B + \bar{C})$$

assuming A=1, B=0, C=1

$$X = \overline{(1 + 0 \cdot 1)} + 1 \cdot (0 + \bar{1})$$

$$X = \overline{(1 + 0)} + 1 \cdot (0 + 0)$$

$$X = \overline{(1)} + 1 \cdot (0)$$

$$X = 0 + 0$$

$$X = 0$$

Figure 2.2 A Boolean Equation

The equations can be manipulated using the basic axioms of Boolean shown in Figure 2.3. A few of the axioms (associative, distributive, commutative) behave like normal algebra, but the other axioms have subtle differences that must not be ignored.

Idempotent

$$A + A = A \qquad\qquad A \cdot A = A$$

Associative

$$(A + B) + C = A + (B + C) \qquad\qquad (A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Commutative

$$A + B = B + A \qquad\qquad A \cdot B = B \cdot A$$

Distributive

$$A + (B \cdot C) = (A + B) \cdot (A + C) \qquad\qquad A \cdot (B + C) = (A \cdot B) + (A \cdot C)$$

Identity

$$A + 0 = A \qquad\qquad A + 1 = 1$$

$$A \cdot 0 = 0 \qquad\qquad A \cdot 1 = A$$

Complement

$$A + \bar{A} = 1 \qquad\qquad \overline{(\bar{A})} = A$$

$$A \cdot \bar{A} = 0 \qquad\qquad \bar{1} = 0$$

DeMorgan's

$$\overline{(A + B)} = \bar{A} \cdot \bar{B} \qquad\qquad \overline{(A \cdot B)} = \bar{A} + \bar{B}$$

Duality

interchange AND and OR operators, as well as all Universal, and Null sets. The resulting equation is equivalent to the original.

Figure 2.3 The Basic Axioms of Boolean Algebra

An example of equation manipulation is shown in Figure 2.4. The distributive axiom is applied to get equation (1). The idempotent axiom is used to get equation (2). Equation (3) is obtained by using the distributive axiom to move C outside the parentheses, but the identity axiom is used to deal with the lone C. The identity axiom is then used to simplify the contents of the parentheses to get equation (4). Finally the Identity axiom is used to get the final, simplified equation. Notice that using Boolean algebra has shown that 3 of the variables are entirely unneeded.

$$A = \overline{B} \cdot (C \cdot (\overline{D} + E + C) + \overline{F} \cdot C)$$

$$A = \overline{B} \cdot (\overline{D} \cdot C + E \cdot C + C \cdot C + \overline{F} \cdot C) \quad (1)$$

$$A = \overline{B} \cdot (\overline{D} \cdot C + E \cdot C + C + \overline{F} \cdot C) \quad (2)$$

$$A = \overline{B} \cdot C \cdot (\overline{D} + E + 1 + \overline{F}) \quad (3)$$

$$A = \overline{B} \cdot C \cdot (1) \quad (4)$$

$$A = \overline{B} \cdot C \quad (5)$$

Figure 2.4 Simplification of a Boolean Equation

Note: When simplifying Boolean algebra, OR operators have a lower priority, so they should be manipulated first. NOT operators have the highest priority, so they should be simplified last. Consider the example from before.

$$X = \overline{(A + B \cdot C)} + A \cdot (B + \overline{C})$$

$$X = \overline{(A) + (B \cdot C)} + A \cdot (B + \overline{C})$$ The higher priority operators are put in parenthases

$$X = \overline{(A)} \cdot \overline{(B \cdot C)} + A \cdot (B + \overline{C})$$ DeMorgan's theorem is applied

$$X = \overline{A} \cdot (\overline{B} + \overline{C}) + A \cdot (B + \overline{C})$$ DeMorgan's theorem is applied again

$$X = \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{C} + A \cdot B + A \cdot \overline{C}$$ The equation is expanded

$$X = \overline{A} \cdot \overline{B} + (\overline{A} \cdot \overline{C} + A \cdot \overline{C}) + A \cdot B$$ Terms with common terms are collected, here it is only NOT C

$$X = \overline{A} \cdot \overline{B} + \overline{C} \cdot (\overline{A} + A) + A \cdot B$$ The redundant term is eliminated

$$X = \overline{A} \cdot \overline{B} + \overline{C} + A \cdot B$$ A Boolean axiom is applied to simplify the equation further

Note: A simplified expression will generally reduce the number of operations required....

$$A + B\overline{A} = A + B$$

2.3 Logic Design

Design ideas can be converted to Boolean equations directly, or with other techniques discussed later. The Boolean equation form can then be simplified or rearranges, and then converted into ladder logic, or a circuit.

If we can describe how a controller should work in words, we can often convert it directly to a Boolean equation, as shown in Figure 2.5. In the example a process description is given first. In actual applications this is obtained by talking to the designer of the mechanical part of the system. In many cases the system does not exist yet, making this a challenging task. The next step is to determine how the controller should work. In this case it is written out in a sentence first, and then converted to a Boolean expression. The Boolean expression may then be converted to a desired form. The first equation contains an EOR, which is not available in ladder logic, so the next line converts this to an equivalent expression (2) using ANDs, ORs and NOTs. The circuit shown is for the second equation. In the conversion the terms that are ANDed are in series.......... The last equation (3) is fully expanded and the circuit for it is shown in Figure 2.6. This illustrates the same logical control function can be achieved with different, yet equivalent, circuits.

Process Description:

A heating oven with two bays can heat one ingot in each bay. When the heater is on it provides enough heat for two ingots. But, if only one ingot is present the oven may become too hot, so a fan is used to cool the oven when it passes a set temperature.

Control Description:

If the temperature is too high and there is an ingot in only one bay then turn on fan.

Define Inputs and Outputs:

B1 = bay 1 ingot present
B2 = bay 2 ingot present
F = fan
T = temperature overheat sensor

Boolean Equation:

$$F = T \cdot (B_1 \oplus B_2)$$

$$F = T \cdot (B_1 \cdot \overline{B_2} + \overline{B_1} \cdot B_2) \qquad (2)$$

$$F = B_1 \cdot \overline{B_2} \cdot T + \overline{B_1} \cdot B_2 \cdot T \qquad (3)$$

Circuit for Equation (2):

XXXXXXX

Note: the result for conditional logic is a single step in the ladder

Warning: in spoken and written english OR and EOR are often not clearly defined. Consider the traffic directions "Go to main street then turn left or right." Does this *or* mean that you can drive either way, or that the person isn't sure which way to go? Consider the expression "The cars are red or blue.", Does this mean that the cars can be either red or blue, or all of the cars are red, or all of the cars are blue. A good literal way to describe this condition is "one or the other, but not both".

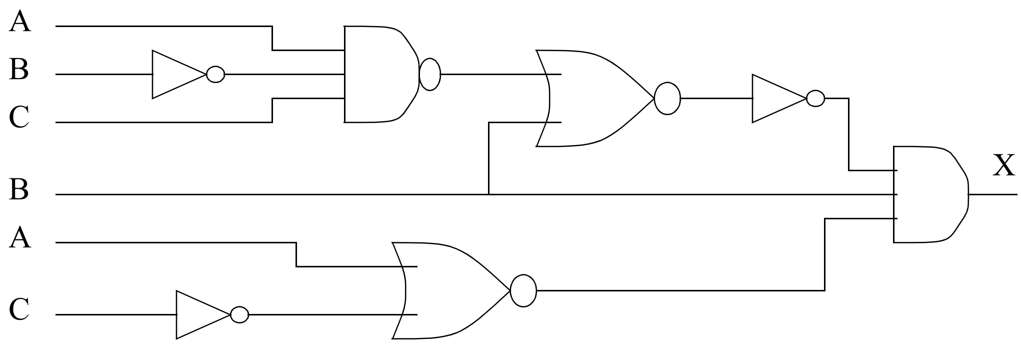
Figure 2.5 Boolean Algebra Based Design of Ladder Logic

Circuit for Equation (3):

XXXXXXXXXXXXXXXX

Figure 2.6 Alternate Circuit

Boolean algebra is often used in the design of digital circuits. Consider the example in Figure 2.7. In this case we are presented with a circuit that is built with inverters, nand, nor and, and gates. This figure can be converted into a boolean equation by starting at the left hand side and working right. Gates on the left hand side are *solved* first, so they are put inside parentheses to indicate priority. Inverters are represented by putting a NOT operator on a variable in the equation. This circuit can't be directly converted to ladder logic because there are no equivalents to NAND and NOR gates. After the circuit is converted to a Boolean equation it is simplified, and then converted back into a (much simpler) circuit diagram.



The circuit is converted to a Boolean equation and simplified. The most nested terms in the equation are on the left hand side of the diagram.

$$X = \overline{\overline{((\overline{A \cdot \overline{B} \cdot C}) + B)}} \cdot B \cdot \overline{(A + \overline{C})}$$

$$X = (\overline{A} + B + \overline{C} + B) \cdot B \cdot (\overline{A} \cdot C)$$

$$X = \overline{A} \cdot B \cdot \overline{A} \cdot C + B \cdot B \cdot \overline{A} \cdot C + \overline{C} \cdot B \cdot \overline{A} \cdot C + B \cdot B \cdot \overline{A} \cdot C$$

$$X = B \cdot \overline{A} \cdot C + B \cdot \overline{A} \cdot C + 0 + B \cdot \overline{A} \cdot C$$

$$X = B \cdot \overline{A} \cdot C$$

This simplified equation is converted back into a circuit.

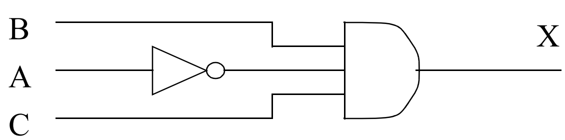


Figure 2.7 Reverse Engineering of a Digital Circuit

To summarize, we will obtain Boolean equations from a verbal description or existing circuit or ladder diagram. The equation can be manipulated using the axioms of Boolean algebra. after simplification the equation can be converted back into a circuit diagram. Circuits can behave the same even though they are in different forms. When simplifying Boolean equations that are to be implemented in circuits there are a few basic rules.

1. Eliminate NOTs that are for more than one variable. This normally includes replacing NAND and NOR functions with simpler ones using DeMorgan's theorem.

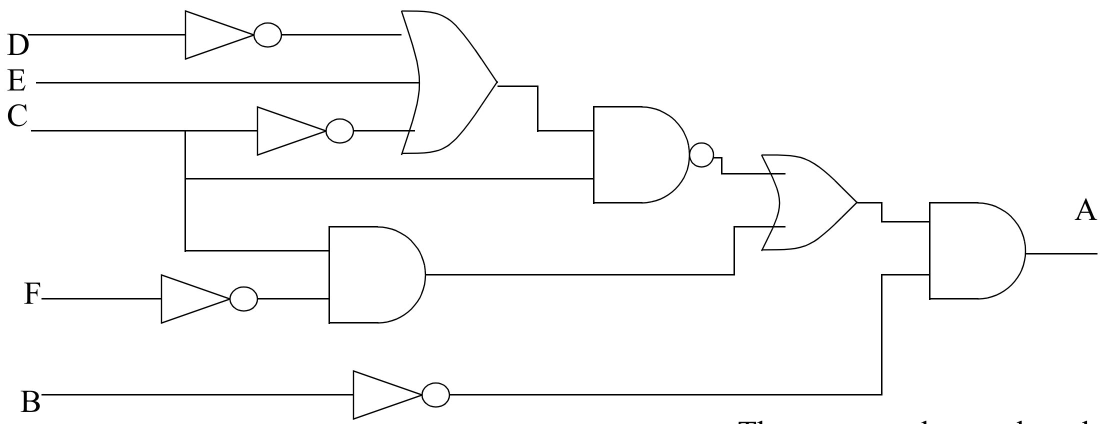
2. Eliminate complex functions such as EORs with their equivalent.

These principles are reinforced with another design that begins in Figure 2.8. Assume that the Boolean equation that describes the controller is already known. This equation can be converted into both a circuit diagram and ladder logic. The circuit diagram contains about two dollars worth of integrated circuits. If the design was mass produced the final cost for the entire controller would be under $50. The prototype of the controller would cost thousands of dollars.

Given the controller equation;

$$A = \overline{B} \cdot (\overline{C \cdot (\overline{D} + E + \overline{C})} + \overline{F} \cdot C)$$

The circuit is given below, and equivalent ladder logic is shown.



The gates can be purchased for about $0.25 each in bulk. Inputs and outputs are typically 5V

Figure 2.8 A Boolean Equation and Derived Circuit

The initial equation is not the simplest. It is possible to simplify the equation to the form seen in Figure 2.8.

$$A = \overline{B} \cdot C \cdot (\overline{D} + E + \overline{F})$$

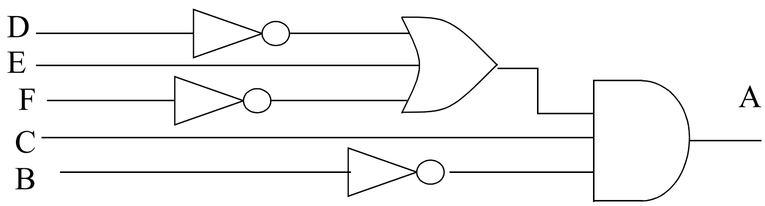


Figure 2.9 The Simplified Form of the Example

The equation can also be manipulated to other forms that are more routine but less efficient as shown in Figure 2.10. The equation shown is in disjunctive normal form - in simpler words this is ANDed terms ORed together. This is also an example of a canonical form - in simpler terms this means a standard form. This form is more important for digital logic. For example, when an equation is simplified, it may not look like the original design intention, and therefore becomes harder to rework without starting from the beginning.

$$A = (\overline{B} \cdot C \cdot \overline{D}) + (\overline{B} \cdot C \cdot E) + (\overline{B} \cdot C \cdot \overline{F})$$

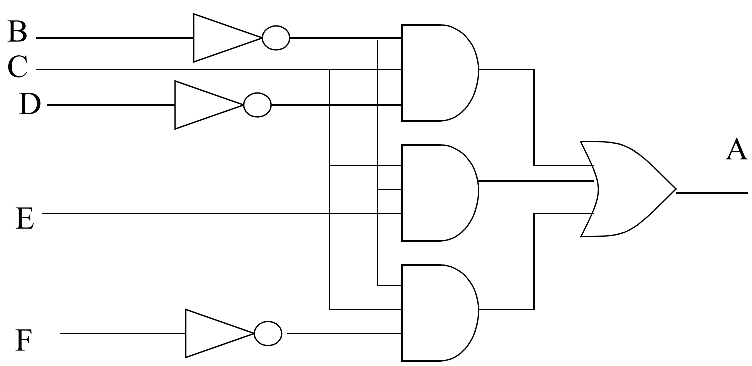


Figure 2.10 A Canonical Logic Form

2.3.1 Boolean Algebra Techniques

There are some common Boolean algebra techniques that are used when simplifying equations. Recognizing these forms are important to simplifying Boolean Algebra with ease. These are itemized, with proofs in Figure 2.11.

$A + C\bar{A} = A + C$ proof:

$$
\begin{aligned}
&A + C\bar{A}\\
&(A + C)(A + \bar{A})\\
&(A + C)(1)\\
&A + C
\end{aligned}
$$

$AB + A = A$ proof:

$$
\begin{aligned}
&AB + A\\
&AB + A1\\
&A(B + 1)\\
&A(1)\\
&A
\end{aligned}
$$

$\overline{A + B + C} = \bar{A}\bar{B}\bar{C}$ proof:

$$
\begin{aligned}
&\overline{A + B + C}\\
&\overline{(A + B) + C}\\
&\overline{(A + B)}\bar{C}\\
&(\bar{A}\bar{B})\bar{C}\\
&\bar{A}\bar{B}\bar{C}
\end{aligned}
$$

Figure 2.11 Common Boolean Algebra Techniques

2.4 Common Logic Forms

Knowing a simple set of logic forms will support a designer when categorizing control problems. The following forms are provided to be used directly, or provide ideas when designing.

2.4.1 Complex Gate Forms

In total there are 16 different possible types of 2-input logic gates. The simplest are AND and OR, the other gates we will refer to as *complex* to differentiate. The three popular complex gates that have been discussed before are NAND, NOR and EOR. All of these can be reduced to simpler forms with only ANDs and ORs, as shown in Figure 2.12.

NAND

$$X = \overline{A \cdot B}$$

$$X = \bar{A} + \bar{B}$$

NOR

$$X = \overline{A + B}$$

$$X = \bar{A} \cdot \bar{B}$$

EOR

$$X = A \oplus B$$

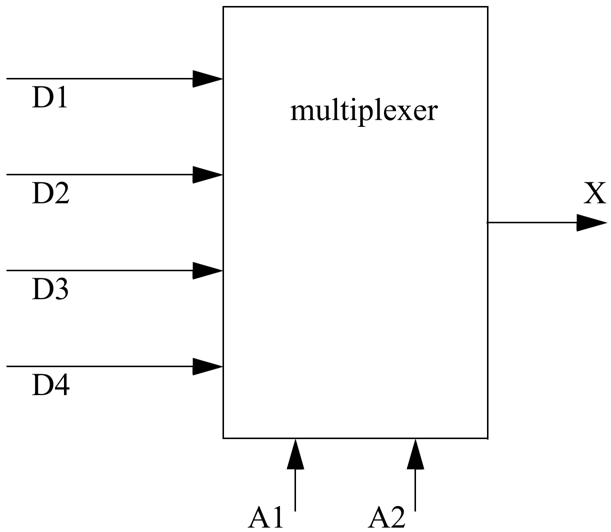
$$X = A \cdot \bar{B} + \bar{A} \cdot B$$

Figure 2.12 Conversion of Complex Logic Functions

2.4.2 Multiplexers

Multiplexers allow multiple devices to be connected to a single device. These are very popular for telephone systems. A telephone *switch* is used to determine which telephone will be connected to a limited number of lines to other telephone switches. This allows telephone calls to be made to somebody far away without a dedicated wire to the other telephone. In older telephone switch boards, operators physically connected wires by plugging them in. In modern computerized telephone switches the same thing is done, but to digital voice signals.

In Figure 2.13 a multiplexer is shown that will take one of four inputs bits D1, D2, D3 or D4 and make it the output X, depending upon the values of the address bits, A1 and A2.



A1	A2	X
0	0	X=D1
0	1	X=D2
1	0	X=D3
1	1	X=D4

Figure 2.13 A Multiplexer

2.5 Simple Design Cases

The following cases are presented to illustrate various combinatorial logic problems, and possible solutions. It is recommended that you try to satisfy the description before looking at the solution.

2.5.1 Basic Logic Functions

Problem: Develop a program that will cause output D to go true when switch A and switch B are closed or when switch C is closed.

Solution:

$$D = (A \cdot B) + C$$

Figure 2.14 Sample Solution for Logic Case Study A

Problem: Develop a program that will cause output D to be on when push button A is on, or either B or C are on.

Solution:

$$D = A + (B \oplus C)$$

Figure 2.15 Sample Solution for Logic Case Study B

2.5.2 Car Safety System

Problem: Develop a circuit for a car door/seat belt safety system. When the car door is open, and the seatbelt is not done up, the ignition power must not be applied. If all is safe then the key will start the engine.

Solution:

Figure 2.16 Solution to Car Safety System Case

2.5.3 Motor Forward/Reverse

Problem: Design a motor controller that has a forward and a reverse button. The motor forward and reverse outputs will only be on when one of the buttons is pushed. When both buttons are pushed the motor will not work.

Solution:

$$F = BF \cdot \overline{BR}$$

$$R = \overline{BF} \cdot BR$$

where,

F = motor forward
R = motor reverse
BF = forward button
BR = reverse button

Figure 2.17 Motor Forward, Reverse Case Study

2.5.4 A Burglar Alarm

Consider the design of a burglar alarm for a house. When activated an alarm and lights will be activated to encourage the unwanted guest to leave. This alarm be activated if an unauthorized intruder is detected by window sensor and a motion detector. The window sensor is effectively a loop of wire that is a piece of thin metal foil that encircles the window. If the window is broken, the foil breaks breaking the conductor. This behaves like a normally closed switch. The motion sensor is designed so that when a person is detected the output will go on. As with any alarm an activate/deactivate switch is also needed. The basic operation of the alarm system, and the inputs and outputs of the controller are itemized in Figure 2.18.

The inputs and outputs are chosen to be;

A = Alarm and lights switch (1 = on)
W = Window/Door sensor (1 = OK)
M = Motion Sensor (0 = OK)
S = Alarm Active switch (1 = on)

The basic operation of the alarm can be described with rules.

1. If alarm is on, check sensors.
2. If window/door sensor is broken (turns off), sound alarm and turn on lights

Note: As the engineer, it is your responsibility to define these items before starting the work. If you do not do this first you are guaranteed to produce a poor design. It is important to develop a good list of inputs and outputs, and give them simple names so that they are easy to refer to. Most companies will use wire numbering schemes on their diagrams.

Figure 2.18 Controller Requirements List for Alarm

The next step is to define the controller equation. In this case the controller has 3 different inputs, and a single output, so a truth table is a reasonable approach to formalizing the system. A Boolean equation can then be written using the truth table in Figure 2.19. Of the eight possible combinations of alarm inputs, only three lead to alarm conditions.

Inputs			Output
S	M	W	A
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

alarm off

alarm on/no thief

alarm on/thief detected

note the binary sequence

Figure 2.19 Truth Table for the Alarm

The Boolean equation in Figure 2.20 is written by examining the truth table in Figure 2.19. There are three possible alarm conditions that can be represented by the conditions of all three inputs. For example take the last line in the truth table where when all three inputs are on the alarm should be one. This leads to the last term in the equation. The other two terms are developed the same way. After the equation has been written, it is simplified.

$$A = (S \cdot \overline{M} \cdot \overline{W}) + (S \cdot M \cdot \overline{W}) + (S \cdot M \cdot W)$$
$$\therefore A = S \cdot (\overline{M} \cdot \overline{W} + M \cdot \overline{W} + M \cdot W)$$
$$\therefore A = S \cdot ((\overline{M} \cdot \overline{W} + M \cdot \overline{W}) + (M \cdot \overline{W} + M \cdot W))$$
$$\therefore A = (S \cdot \overline{W}) + (S \cdot M) = S \cdot (\overline{W} + M)$$

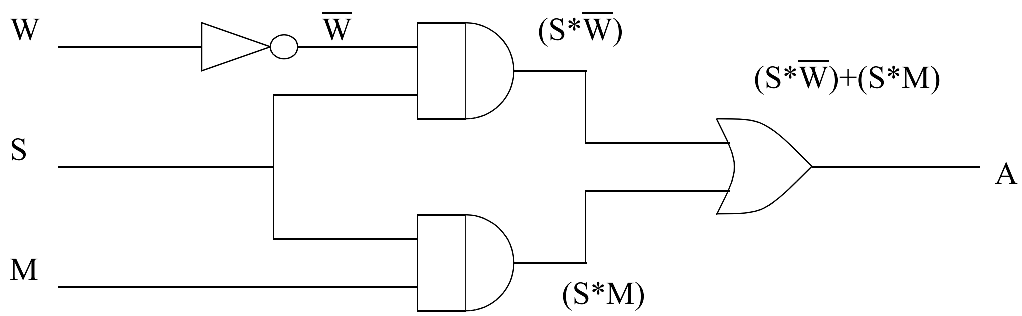


Figure 2.20 A Boolean Equation and Implementation for the Alarm

The equation and circuits shown in Figure can also be further simplified, as shown in Figure 2.21.

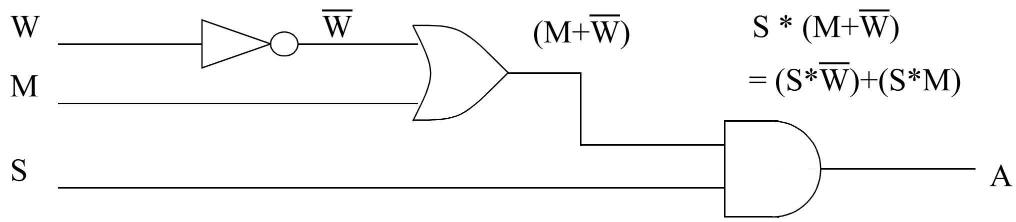


Figure 2.21 The Simplest Circuit and Ladder Diagram

Aside: The alarm could also be implemented in programming languages. The program below is for a Basic Stamp II chip. (www.parallaxinc.com)

```
w = 1; s = 2; m = 3; a = 4
input m; input w; input s
output a
loop:
if (in2 = 1) and (in1 = 0 or in3 = 1) then on
low a; goto loop 'alarm off
on:
high a; goto loop 'alarm on
```

Figure 2.22 Alarm Implementation Using A High Level Programming Language

2.6 Summary

- Logic can be represented with Boolean equations.
- Boolean equations can be converted to (and from) digital circuits.
- Boolean equations can be simplified.
- Different controllers can behave the same way.
- Common logic forms exist and can be used to understand logic.

• Truth tables can represent all of the possible state of a system.

2.7 Problems

1. Draw a circuit that will cause output D to go true when switch A and switch B are closed or when switch C is closed.

2. Draw a circuit that will cause output D to be on when push button A is on, or either B or C are on.

3. Design a circuit for a car that considers the variables below to control the motor *M*. Also add a second output that uses any outputs not used for motor control.

   - doors opened/closed (D)
   - keys in ignition (K)
   - motor running (M)
   - transmission in park (P)
   - ignition start (I)

4. Make a simple circuit that will turn on the outputs with the binary patterns when the corresponding buttons are pushed. Inputs X, Y, and Z will never be on at the same time.

OUTPUTS								INPUTS
H	G	F	E	D	C	B	A	
1	1	0	1	0	1	0	1	Input X on
1	0	1	0	0	0	0	1	Input Y on
1	0	0	1	0	1	1	1	Input Z on

5. Convert the following Boolean equation to the simplest possible circuit.

$$X = A \cdot \overline{(\bar{A} + \bar{A} \cdot B)}$$

6. Simplify the following boolean equations.

a) $A(B + AB)$

b) $\overline{A(\overline{B + AB})}$

c) $\bar{A}(B + AB)$

d) $\overline{\bar{A}(\overline{B + AB})}$

7. Simplify the following Boolean equations,

a) $\overline{(A+B)} \cdot \overline{(A+\overline{B})}$

b) $ABCD + \overline{A}BCD + ABC\overline{D} + AB\overline{C}\overline{D}$

8. Simplify the Boolean expression below.

$$((A \cdot \overline{B}) + \overline{(\overline{B}+A)}) \cdot C + (\overline{B} \cdot C + B \cdot C)$$

9. Given the Boolean expression a) draw a digital circuit and b) simplify the expression.

$$X = A \cdot \overline{B} \cdot C + \overline{(C+B)}$$

10. Simplify the following Boolean equation and write a corresponding circuit.

$$Y = \overline{\overline{(AB\overline{C}D + AB\overline{C}\overline{D} + \overline{A}BCD + \overline{A}\overline{B}CD)} + D}$$

11. For the following Boolean equation,

$$X = A + B(A + C\overline{B} + D\overline{A}C) + ABCD$$

a) Write the logic circuit for the unsimplified equation.
b) Simplify the equation.
c) Write the circuit for the simplified equation.

12. a) Write a Boolean equation for the following truth table. (Hint: do this by writing an expres-

sion for each line with a true output, and then ORing them together.)

A	B	C	D	Result
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	1

b) Write the results in a) in a Boolean equation.
c) Simplify the Boolean equation in b)

13. Simplify the following Boolean equation, and create the simplest circuit.

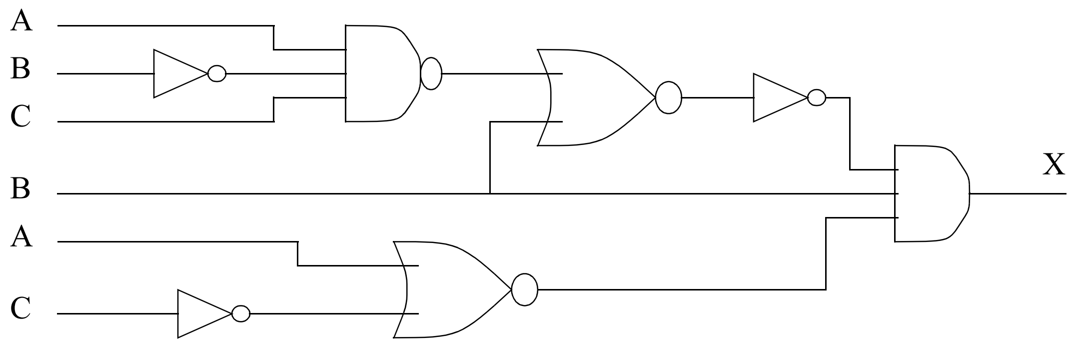
$$Y = \overline{\bar{C}\left(\overline{\bar{A} + \left(\overline{\bar{A} + \overline{(\bar{B}\bar{C}(\overline{A + \overline{B\bar{C}}}))}}\right)}\right)}$$

14. Simplify the following boolean equation with Boolean algebra and write the corresponding circuit.

$$X = \overline{(A + B \cdot \bar{A}) + \overline{(C + D + E\bar{C})}}$$

15. a) Develop the Boolean expression for the circuit below.
b) Simplify the Boolean expression.

c) Draw a simpler circuit for the equation in b).



16. Given a system that is described with the following equation,

$$X = A + (B \cdot (\bar{A} + C) + C) + A \cdot B \cdot (\bar{D} + \bar{E})$$

a) Simplify the equation using Boolean Algebra.
b) Implement the original and then the simplified equation with a digital circuit.

17. Simplify the following and implement the original and simplified equations with gates.

$$A + (\bar{B} + \bar{C} + \bar{D}) \cdot (B + \bar{C}) + A \cdot B \cdot (\bar{C} + \bar{D})$$

18. Simplify the following Boolean equation and implement it in a circuit.

$$X = A + BA + B\bar{C} + \overline{D + C}$$

19. Use Boolean equations to develop simplified circuit for the following truth table where A, B, C and D are inputs, and X and Y are outputs.

A	B	C	D	X	Y
0	0	0	0	0	0
0	0	0	1	1	0
0	0	1	0	0	0
0	0	1	1	1	0
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
0	1	1	1	0	1
1	0	0	0	0	0
1	0	0	1	1	0
1	0	1	0	0	0
1	0	1	1	1	0
1	1	0	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1
1	1	1	1	1	1

20. Convert the truth table below to a Boolean equation, and then simplify it. The output is X and the inputs are A, B, C and D.

A	B	C	D	X
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

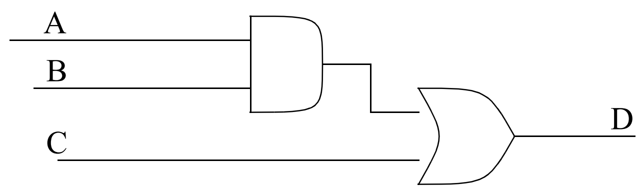
21. Simplify the following Boolean equation. Convert both the unsimplified and simplified equa-

tions to a circuit.

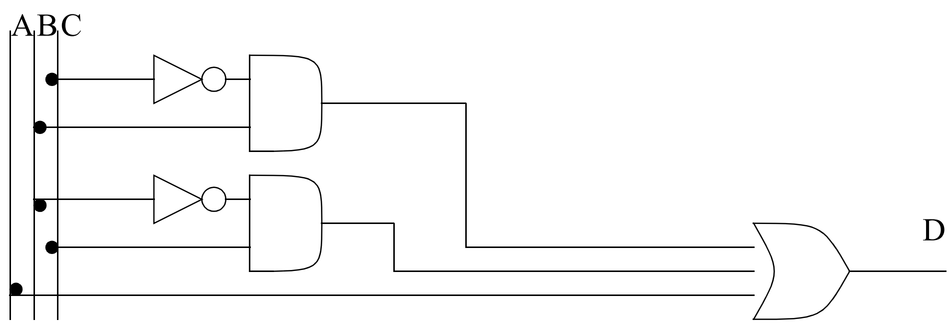
$$X = \overline{(AB\overline{C})}(A + BC)$$

2.8 Problem Solutions

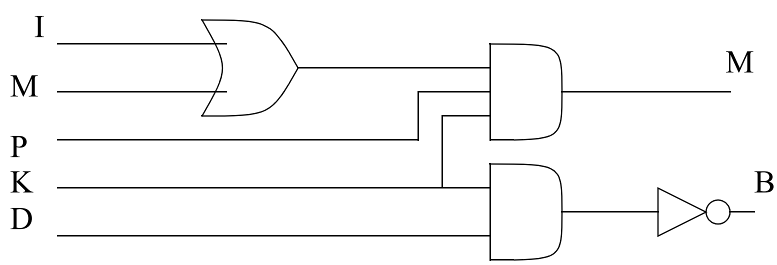
1.



2.



3.



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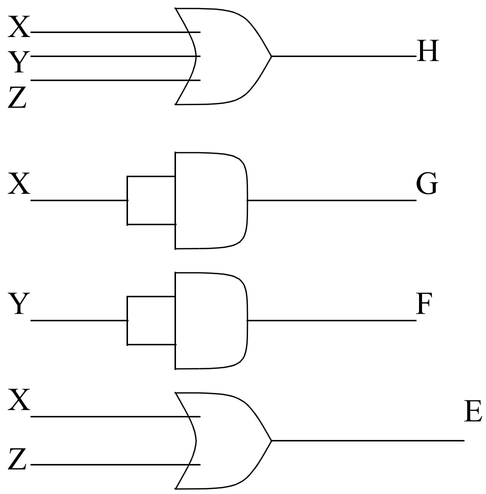

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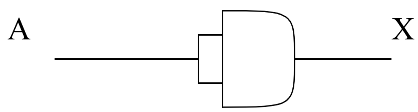


6.

a) AB b) $\bar{A}+B$ c) $\bar{A}B$ d) $A+B$

7.

a) $$\overline{(A+B)}\cdot\overline{(A+\bar{B})} = (\bar{A}\bar{B})(\bar{A}B) = 0$$

b) $$ABCD+\bar{A}BCD+ABC\bar{D}+AB\bar{C}\bar{D} = BCD+AB\bar{D} = B(CD+A\bar{D})$$

8. C

9.

$$X = \bar{B}\cdot(A\cdot C+\bar{C})$$

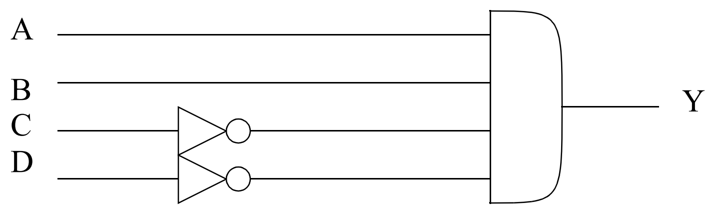
10.

$$Y = \overline{\overline{(AB\overline{C}D + AB\overline{C}\overline{D} + \overline{A}BCD + \overline{A}\overline{B}CD)} + D}$$

$$Y = (AB\overline{C}D + AB\overline{C}\overline{D} + \overline{A}BCD + \overline{A}\overline{B}CD)\overline{D}$$

$$Y = (0 + AB\overline{C}\overline{D} + 0 + 0)\overline{D}$$

$$Y = AB\overline{C}\overline{D}$$

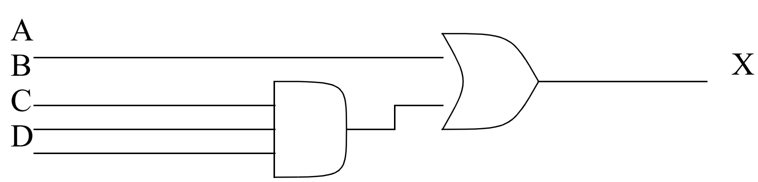


11.

a)

b) $X = A + DCB$

c)



12.

$$\bar{A}\bar{B}\bar{C}\bar{D}+\bar{A}\bar{B}CD+\bar{A}B\bar{C}D+\bar{A}BCD+A\bar{B}\bar{C}\bar{D}+A\bar{B}CD+ABC\bar{D}+ABCD$$

$$\bar{B}\bar{C}\bar{D}+\bar{A}CD+\bar{B}CD+\bar{A}BD+BCD+ACD+ABC$$

$$\bar{B}\bar{C}\bar{D}+CD(\bar{A}+A)+CD(\bar{B}+B)+\bar{A}BD+ABC$$

$$\bar{B}\bar{C}\bar{D}+D(C+\bar{A}B)+ABC$$

13.

$$Y = \overline{\bar{C}\left(\overline{\bar{A}+\left(\overline{\bar{A}+\overline{(\bar{B}\bar{C}\overline{(A+\overline{B\bar{C}})})}}\right)}\right)}$$

$$Y = \overline{\bar{C}\left(\overline{\bar{A}+\left(\overline{\bar{A}+\overline{(\bar{B}\bar{C}\overline{(A+\bar{B}+C)})}}\right)}\right)}$$

$$Y = \overline{\bar{C}\left(\overline{\bar{A}+(\overline{\bar{A}+\overline{(\bar{B}\bar{C}\bar{A}B\bar{C})}})}\right)}$$

$$Y = \overline{\bar{C}\left(\overline{\bar{A}+(\overline{\bar{A}+\bar{0}})}\right)}$$

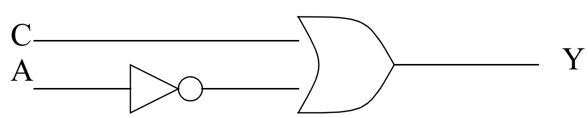
$$Y = \overline{\bar{C}(\overline{\bar{A}+(\overline{\bar{A}+1})})}$$

$$Y = \overline{\bar{C}(\overline{\bar{A}+(\bar{1})})}$$

$$Y = \overline{\bar{C}(\overline{\bar{A}+0})}$$

$$Y = \overline{\bar{C}A}$$

$$Y = C+\bar{A}$$



14.

$$X = \overline{(A + B \cdot \bar{A}) + \overline{(C + D + E\bar{C})}}$$

$$X = \overline{(A + B \cdot \bar{A})}(C + D + E\bar{C})$$

$$X = (\bar{A})(\overline{B \cdot \bar{A}})(C + D + E\bar{C})$$

$$X = (\bar{A})(\overline{B \cdot \bar{A}})(C + D + E\bar{C})$$

$$X = \bar{A}\bar{B}(C + D + E\bar{C})$$

$$X = \bar{A}\bar{B}(C + D + E)$$

OR

$$X = \overline{(A + B \cdot \bar{A}) + \overline{(C + D + E\bar{C})}}$$

$$X = \overline{A + B \cdot \bar{A} + \bar{C}\bar{D}(\bar{E} + C)}$$

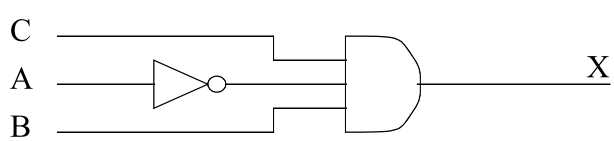
$$X = \overline{A + B + \bar{C}\bar{D}\bar{E}}$$

$$X = \bar{A}\bar{B}(\overline{\bar{C}\bar{D}\bar{E}})$$

$$X = \bar{A}\bar{B}(C + D + E)$$

15.

$C\bar{A}B$



16.

a)

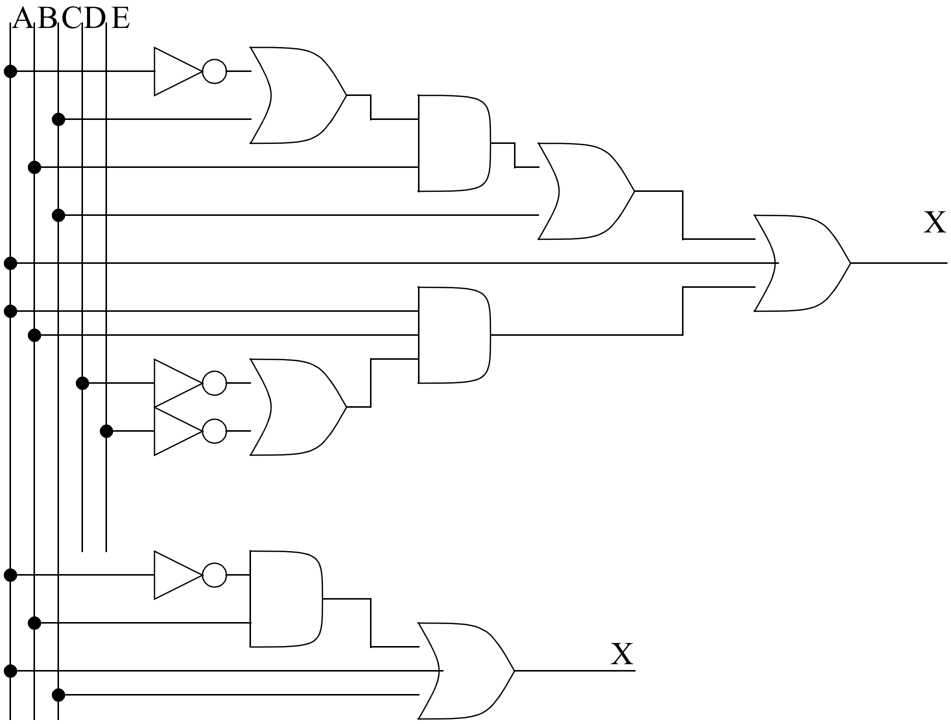
$$X = A + (B \cdot (\overline{A} + C) + C) + A \cdot B \cdot (\overline{D} + \overline{E})$$

$$X = A + (B \cdot \overline{A} + B \cdot C + C) + A \cdot B \cdot \overline{D} + A \cdot B \cdot \overline{E}$$

$$X = A \cdot (1 + B \cdot \overline{D} + B \cdot \overline{E}) + B \cdot \overline{A} + C \cdot (B + 1)$$

$$X = A + B \cdot \overline{A} + C$$

b)



17.

$$A + (\overline{B} + \overline{C} + \overline{D}) \cdot (B + \overline{C}) + A \cdot B \cdot (\overline{C} + \overline{D})$$

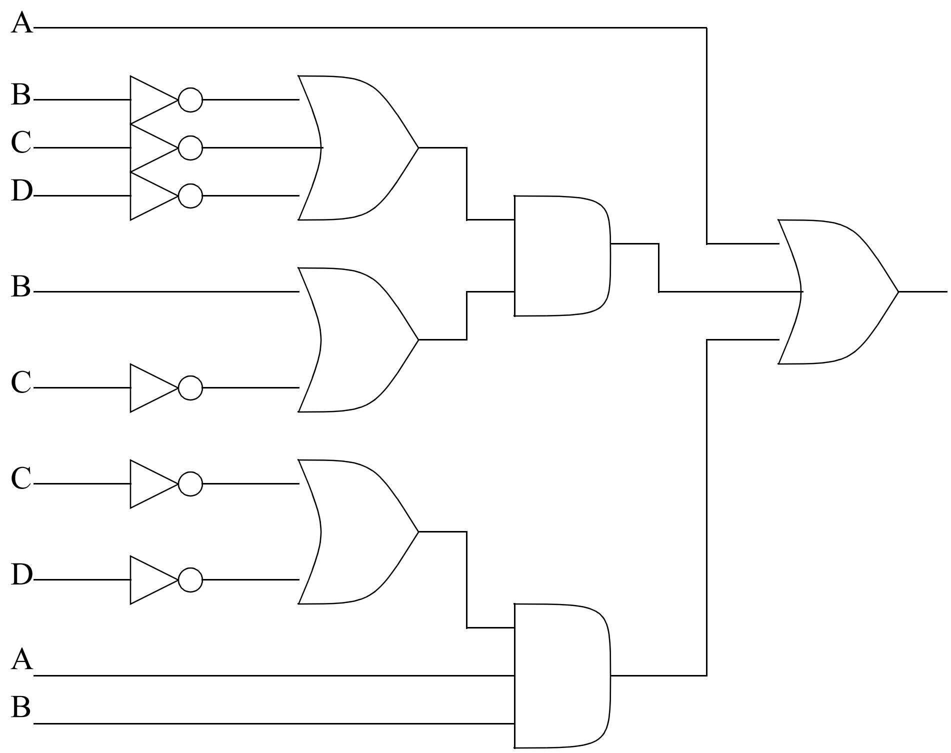
$$A \cdot (1 + B \cdot (\overline{C} + \overline{D})) + (\overline{B} + \overline{C} + \overline{D}) \cdot B + (\overline{B} + \overline{C} + \overline{D}) \cdot \overline{C}$$

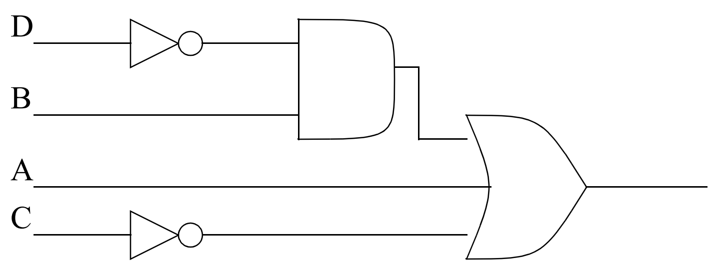
$$A + (\overline{C} + \overline{D}) \cdot B + \overline{C}$$

$$A + \overline{C} \cdot B + \overline{D} \cdot B + \overline{C}$$

$$A + \overline{D} \cdot B + \overline{C}$$

$A + \overline{D} \cdot B + \overline{C}$





2.9 Challenge Problems

1. Write a program that implements the following Boolean expression in Scilab or C. The user should be able input values and the results should be printed.

$$A + (\overline{B} + \overline{C} + \overline{D}) \cdot (B + \overline{C}) + A \cdot B \cdot (\overline{C} + \overline{D})$$

3. NUMBERS AND DATA

Topics:
- Number bases; binary, octal, decimal, hexadecimal
- Binary calculations; 2s compliments, addition, subtraction and Boolean operations
- Encoded values; BCD and ASCII
- Error detection; parity, gray code and checksums

Objectives:
- To be familiar with binary, octal and hexadecimal numbering systems.
- To be able to convert between different numbering systems.
- To understand 2s compliment negative numbers.
- To be able to convert ASCII and BCD values.
- To be aware of basic error detection techniques.

3.1 Introduction

Base 10 (decimal) numbers developed naturally because the original developers (probably) had ten fingers, or 10 digits. Now consider logical systems that only have wires that can be on or off. When counting with a wire the only digits are 0 and 1, giving a base 2 numbering system. Numbering systems for computers are often based on base 2 numbers, but base 4, 8, 16 and 32 are commonly used. A list of numbering systems is give in Figure 3.1. An example of counting in these different numbering systems is shown in Figure 3.2.

Base	Name	Data Unit
2	Binary	Bit
8	Octal	Nibble
10	Decimal	Digit
16	Hexadecimal	Byte

Figure 3.1 Numbering Systems

decimal	binary	octal	hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	a
11	1011	13	b
12	1100	14	c
13	1101	15	d
14	1110	16	e
15	1111	17	f
16	10000	20	10
17	10001	21	11
18	10010	22	12
19	10011	23	13
20	10100	24	14

Note: As with all numbering systems

most significant digits are at left,
least significant digits are at right.

Figure 3.2 Numbers in Decimal, Binary, Octal and Hexadecimal

The effect of changing the base of a number does not change the actual value, only how it is written. The basic rules of mathematics still apply, but many beginners will feel disoriented. This chapter will cover basic topics that are needed to use more complex programming instructions later in the book. These will include the basic number systems, conversion between different number bases, and some data oriented topics.

3.2 Numerical Values

3.2.1 Binary

Binary numbers are the most fundamental numbering system in all computers. A single binary digit (a bit) corresponds to the condition of a single wire. If the voltage on the wire is true the bit value is *1*. If the voltage is off the bit value is *0*. If two or more wires are used then each new wire adds another significant digit. Each binary number will have an equivalent digital value. Figure 3.3 shows how to convert a binary number to a decimal equivalent. Consider the digits, starting at the right. The least significant digit is *1*, and is in the 0th position. To convert this to a decimal equivalent the number base (2) is raised to the position of the digit, and multiplied by the digit. In this case the least significant digit is a trivial conversion. Consider the most significant digit, with a value of *1* in the 6th position. This is converted by the number base to the exponent 6 and multiplying by the digit value of 1. This method can also be used for converting the other number system to decimal.

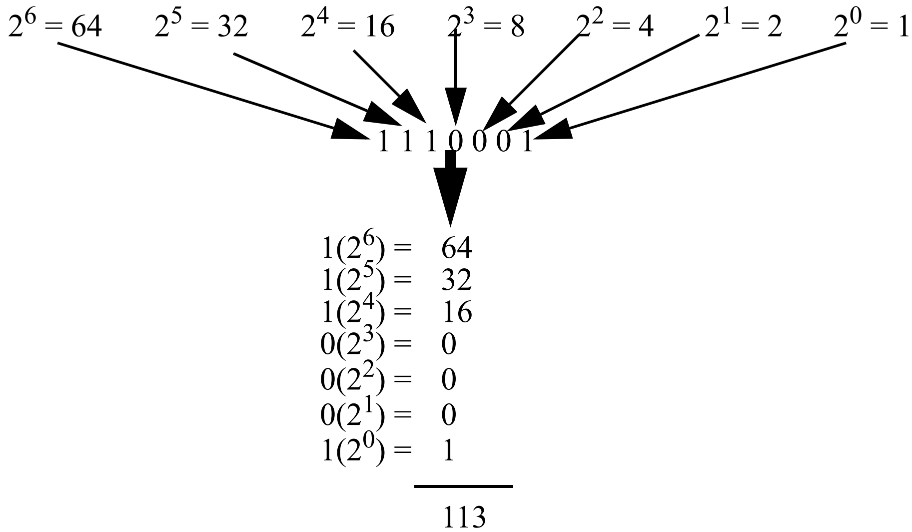
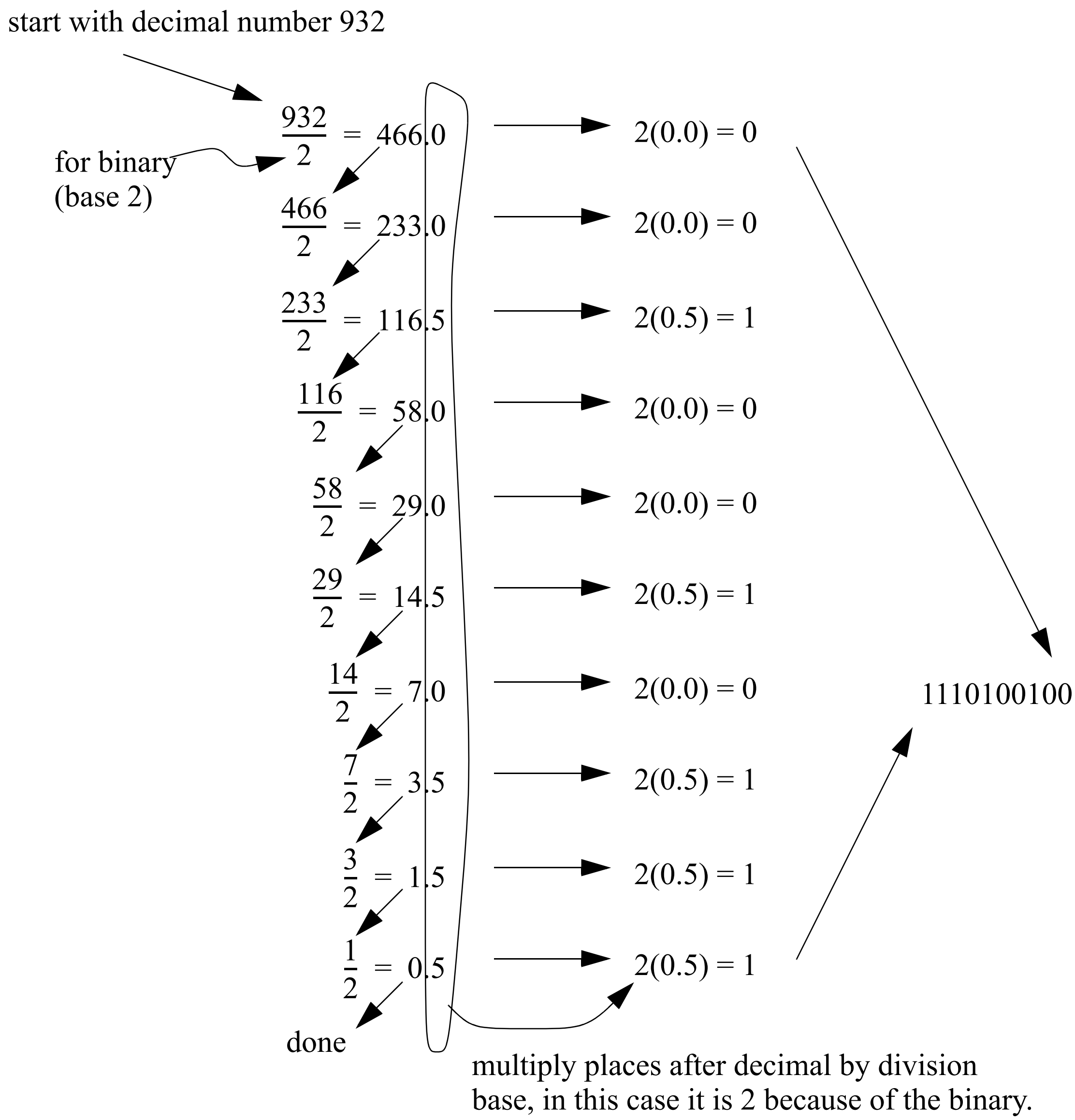


Figure 3.3 Conversion of a Binary Number to a Decimal Number

Decimal numbers can be converted to binary numbers using division, as shown in Figure 3.4. This technique begins by dividing the decimal number by the base of the new number. The fraction after the decimal gives the least significant digit of the new number when it is multiplied by the number base. The whole part of the number is now divided again. This process continues until the whole number is zero. This method will also work for conversion to other number bases.



* This method works for other number bases also, the divisor and multipliers should be changed to the new number bases.

Figure 3.4 Conversion from Decimal to Binary

Most scientific calculators will convert between number bases. But, it is important to understand the conversions between number bases. And, when used frequently enough the conversions can be done in your head.

Binary numbers come in three basic forms - a bit, a byte and a word. A bit is a single binary digit, a byte is eight binary digits, and a word is 16 digits. Words and bytes are shown in Figure 3.5. Notice that on both numbers the least significant digit is on the right hand side of the numbers. And, in the word there are two bytes, and the right hand one is the least significant byte.

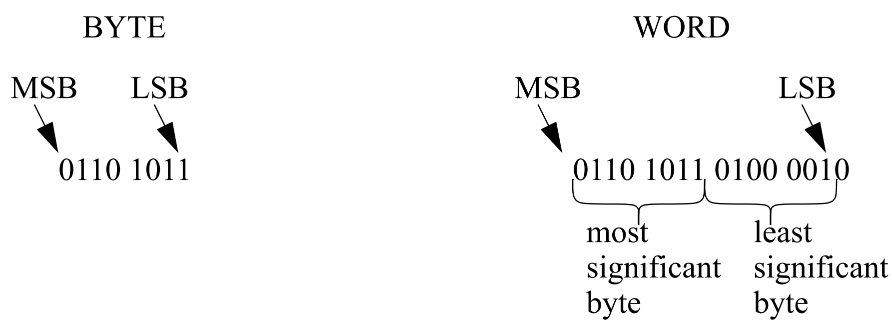


Figure 3.5 Bytes and Words

Binary numbers can also represent fractions, as shown in Figure 3.6. The conversion to and from binary is identical to the previous techniques, except that for values to the right of the decimal the equivalents are fractions.

binary: 101.011

$$1(2^2) = 4 \qquad 0(2^1) = 0 \qquad 1(2^0) = 1 \qquad 0(2^{-1}) = 0 \qquad 1(2^{-2}) = \frac{1}{4} \qquad 1(2^{-3}) = \frac{1}{8}$$

$$= 4 + 0 + 1 + 0 + \frac{1}{4} + \frac{1}{8} = 5.375 \quad \text{decimal}$$

Figure 3.6 A Binary Decimal Number

3.2.1.1 - Boolean Operations

In the next chapter you will learn that entire blocks of inputs and outputs can be used as a single binary number (typically a word). Each bit of the number would correspond to an output or input as shown in Figure 3.7.

There are three motors M_1, M_2 and M_3 represented with three bits in a binary number. When any bit is on the corresponding motor is on.

100 = Motor 1 is the only one on
111 = All three motors are on
in total there are 2^n or 2^3 possible combinations of motors on.

Figure 3.7 Motor Outputs Represented with a Binary Number

We can then manipulate the inputs or outputs using Boolean operations. Boolean algebra has been discussed before for variables with single values, but it is the same for multiple bits. Common operations that use multiple bits in numbers are shown in Figure 3.8. These operations compare only one bit at a time in the number, except the shift instructions that move all the bits one place left or right.

Name	Example	Result	
AND	0010 * 1010	0010	
OR	0010 + 1010	1010	
NOT	$\overline{0010}$	1101	
EOR	0010 eor 1010	1000	
NAND	0010 $\overline{*}$ 1010	1101	
shift left	111000	110001	(other results are possible)
shift right	111000	011100	(other results are possible)
etc.			

Figure 3.8 Boolean Operations on Binary Numbers

3.2.1.2 - Binary Mathematics

Negative numbers are a particular problem with binary numbers. As a result there are three common numbering systems used as shown in Figure 3.9. Unsigned binary numbers are common, but they can only be used for positive values. Both signed and 2s compliment numbers allow positive and negative values, but the maximum positive values is reduced by half. 2s compliment numbers are very popular because the hardware and software to add and subtract is simpler and faster. All three types of numbers will be found in PLCs.

Type	Description	Range for Byte
unsigned	binary numbers can only have positive values.	0 to 255
signed	the most significant bit (MSB) of the binary number is used to indicate positive/negative.	-127 to 127
2s compliment	negative numbers are represented by complimenting the binary number and then adding 1.	-128 to 127

Figure 3.9 Binary (Integer) Number Types

Examples of signed binary numbers are shown in Figure 3.10. These numbers use the most significant bit to indicate when a number is negative.

decimal	binary byte	
2	00000010	
1	00000001	
0	00000000	
-0	10000000	Note: there are two zeros
-1	10000001	
-2	10000010	

Figure 3.10 Signed Binary Numbers

An example of 2s compliment numbers are shown in Figure 3.11. Basically, if the number is positive, it will be a regular binary number. If the number is to be negative, we start the positive number, compliment it (reverse all the bits), then add 1. Basically when these numbers are negative, then the most significant bit is set. To convert from a negative 2s compliment number, subtract 1, and then invert the number.

decimal	binary byte
2	00000010
1	00000001
0	00000000
-1	11111111
-2	11111110

METHOD FOR MAKING A NEGATIVE NUMBER

1. write the binary number for the positive

    for -30 we write 30 = 00011110

2. Invert (compliment) the number

    00011110 becomes 11100001

3. Add 1

    11100001 + 00000001 = 11100010

Figure 3.11 2s Compliment Numbers

Using 2s compliments for negative numbers eliminates the redundant zeros of signed binaries, and makes the hardware and software easier to implement. As a result most of the integer operations in a PLC will do addition and subtraction using 2s compliment numbers. When adding 2s compliment numbers, we don't need to pay special attention to negative values. And, if we want to subtract one number from another, we apply the twos compliment to the value to be subtracted, and then apply it to the other value.

Figure 3.12 shows the addition of numbers using 2s compliment numbers. The three operations result in zero, positive and negative values. Notice that in all three operation the top number is positive, while the bottom operation is negative (this is easy to see because the MSB of the numbers is set). All three of the additions are using bytes, this is important for considering the results of the calculations. In the left and right hand calculations the additions result in a 9th bit - when dealing with 8 bit numbers we call this bit the carry *C*. If the calculation started with a positive and negative value, and ended up with a carry bit, there is no problem, and the carry bit should be ignored. If doing the calculation on a calculator you will see the carry bit, but when using a PLC you must look elsewhere to find it.

```
   00000001 = 1          00000001 = 1           00000010 = 2
+  11111111 = -1      +  11111110 = -2       +  11111111 = -1
  ______________        ______________         ______________
C+00000000 = 0          11111111 = -1        C+00000001 = 1
```

ignore the carry bits

Note: Normally the carry bit is ignored during the operation, but some additional logic is required to make sure that the number has not *overflowed* and moved outside of the range of the numbers. Here the 2s compliment byte can have values from -128 to 127.

Figure 3.12 Adding 2s Compliment Numbers

The integers have limited value ranges, for example a 16 bit word ranges from -32,768 to 32,767. In some cases calculations will give results outside this range, and the Overflow *O* bit will be set. (Note: an overflow condition is a major error, and the PLC will probably halt when this happens.) For an addition operation the Overflow bit will be set when the sign of both numbers is the same, but the sign of the result is opposite. When the signs of the numbers are opposite an overflow cannot occur. This can be seen in Figure 3.13 where the numbers two of the three calculations are outside the range. When this happens the result goes from positive to negative, or the other way.

```
   01111111 = 127        10000001 = -127        10000001 = -127
+  00000011 = 3       +  11111111 = -1       +  11111110 = -2
  ______________        ______________         ______________
   10000010 = -126       10000000 = -128        01111111 =  127
   C = 0                 C = 1                  C = 1
   O = 1 (error)         O = 0 (no error)       O = 1 (error)
```

Note: If an overflow bit is set this indicates that a calculation is outside and acceptable range. When this error occurs the PLC will halt. Do not ignore the limitations of the numbers.

Figure 3.13 Carry and Overflow Bits

These bits also apply to multiplication and division operations. In addition the PLC will also have bits to indicate when the result of an operation is zero *Z* and negative *N*.

3.2.2 Other Base Number Systems

Other number bases are typically converted to and from binary for storage and mathematical operations. Hexadecimal numbers are popular for representing binary values because they are quite compact compared to binary. (Note: large binary numbers with a long string of 1s and 0s are next to impossible to read.) Octal numbers are also popular for inputs and outputs because they work in counts of eight; inputs and outputs are in counts of eight.

An example of conversion to, and from, hexadecimal is shown in Figure 3.14 and Figure 3.15. Note that both of these conversions are identical to the methods used for binary numbers, and the same techniques extend to octal numbers also.

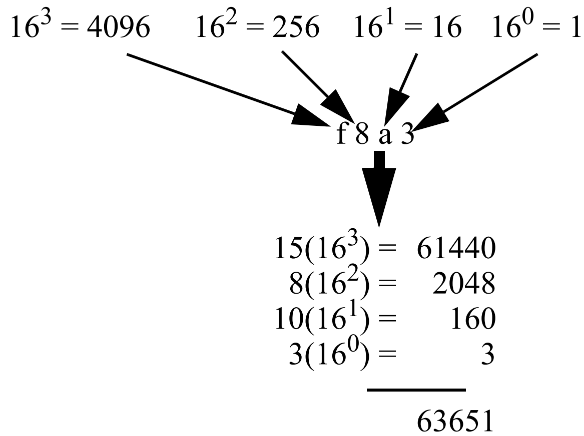


Figure 3.14 Conversion of a Hexadecimal Number to a Decimal Number

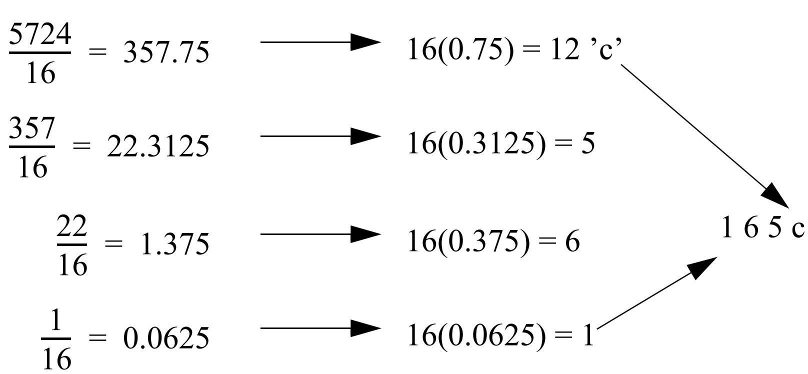


Figure 3.15 Conversion from Decimal to Hexadecimal

3.2.3 BCD (Binary Coded Decimal)

Binary Coded Decimal (BCD) numbers use four binary bits (a nibble) for each digit. (Note: this is not a base number system, but it only represents decimal digits.) This means that one byte can hold two digits from *00* to *99*, whereas in binary it could hold from 0 to 255. A separate bit must be assigned for negative numbers. This method is very popular when numbers are to be output or input to the computer. An example of a BCD number is shown in Figure 3.16. In the example there are four digits, therefore 16 bits are required. Note that the most significant digit and bits are both on the left hand side. The BCD number is the binary equivalent of each digit.

1263 decimal

0001 0010 0110 0011 BCD

Note: this example shows four digits in two bytes. The hex values would also be 1263.

Figure 3.16 A BCD Encoded Number

Most PLCs store BCD numbers in words, allowing values between *0000* and *9999*. They also provide functions to convert to and from BCD. It is also possible to calculations with BCD numbers, but this is uncommon, and when necessary most PLCs have functions to do the calculations. But, when doing calculations you should probably avoid BCD and use integer mathematics instead. Try to be aware when your numbers are BCD values and convert them to *integer* or binary value before doing any calculations.

3.3 Data Characterization

3.3.1 ASCII (American Standard Code for Information Interchange)

When dealing with non-numerical values or data we can use plain text characters and strings. Each character is given a unique identifier and we can use these to store and interpret data. The ASCII (American Standard Code for Information Interchange) is a very common character encryption system is shown in Figure 3.17 and Figure 3.18. The table includes the basic written characters, as well as some special characters, and some control codes. Each one is given a unique number. Consider the letter *A*, it is readily recognized by most computers world-wide when they see the number 65.

decimal	hexadecimal	binary	ASCII	decimal	hexadecimal	binary	ASCII
0	0	00000000	NUL	32	20	00100000	space
1	1	00000001	SOH	33	21	00100001	!
2	2	00000010	STX	34	22	00100010	“
3	3	00000011	ETX	35	23	00100011	#
4	4	00000100	EOT	36	24	00100100	$
5	5	00000101	ENQ	37	25	00100101	%
6	6	00000110	ACK	38	26	00100110	&
7	7	00000111	BEL	39	27	00100111	‘
8	8	00001000	BS	40	28	00101000	(
9	9	00001001	HT	41	29	00101001	)
10	A	00001010	LF	42	2A	00101010	*
11	B	00001011	VT	43	2B	00101011	+
12	C	00001100	FF	44	2C	00101100	,
13	D	00001101	CR	45	2D	00101101	-
14	E	00001110	S0	46	2E	00101110	.
15	F	00001111	S1	47	2F	00101111	/
16	10	00010000	DLE	48	30	00110000	0
17	11	00010001	DC1	49	31	00110001	1
18	12	00010010	DC2	50	32	00110010	2
19	13	00010011	DC3	51	33	00110011	3
20	14	00010100	DC4	52	34	00110100	4
21	15	00010101	NAK	53	35	00110101	5
22	16	00010110	SYN	54	36	00110110	6
23	17	00010111	ETB	55	37	00110111	7
24	18	00011000	CAN	56	38	00111000	8
25	19	00011001	EM	57	39	00111001	9
26	1A	00011010	SUB	58	3A	00111010	:
27	1B	00011011	ESC	59	3B	00111011	;
28	1C	00011100	FS	60	3C	00111100	<
29	1D	00011101	GS	61	3D	00111101	=
30	1E	00011110	RS	62	3E	00111110	>
31	1F	00011111	US	63	3F	00111111	?

Figure 3.17 ASCII Character Table

decimal	hexadecimal	binary	ASCII	decimal	hexadecimal	binary	ASCII
64	40	01000000	@	96	60	01100000	‘
65	41	01000001	A	97	61	01100001	a
66	42	01000010	B	98	62	01100010	b
67	43	01000011	C	99	63	01100011	c
68	44	01000100	D	100	64	01100100	d
69	45	01000101	E	101	65	01100101	e
70	46	01000110	F	102	66	01100110	f
71	47	01000111	G	103	67	01100111	g
72	48	01001000	H	104	68	01101000	h
73	49	01001001	I	105	69	01101001	i
74	4A	01001010	J	106	6A	01101010	j
75	4B	01001011	K	107	6B	01101011	k
76	4C	01001100	L	108	6C	01101100	l
77	4D	01001101	M	109	6D	01101101	m
78	4E	01001110	N	110	6E	01101110	n
79	4F	01001111	O	111	6F	01101111	o
80	50	01010000	P	112	70	01110000	p
81	51	01010001	Q	113	71	01110001	q
82	52	01010010	R	114	72	01110010	r
83	53	01010011	S	115	73	01110011	s
84	54	01010100	T	116	74	01110100	t
85	55	01010101	U	117	75	01110101	u
86	56	01010110	V	118	76	01110110	v
87	57	01010111	W	119	77	01110111	w
88	58	01011000	X	120	78	01111000	x
89	59	01011001	Y	121	79	01111001	y
90	5A	01011010	Z	122	7A	01111010	z
91	5B	01011011	[	123	7B	01111011	{
92	5C	01011100	yen	124	7C	01111100	|
93	5D	01011101	]	125	7D	01111101	}
94	5E	01011110	^	126	7E	01111110	r arr.
95	5F	01011111	_	127	7F	01111111	l arr.

Figure 3.18 ASCII Character Table

This table has the codes from 0 to 127, but there are more extensive tables that contain special graphics symbols, international characters, etc. It is best to use the basic codes, as they are supported widely, and should suffice for all controls tasks.

An example of a string of characters encoded in ASCII is shown in Figure 3.19.

e.g. The sequence of numbers below will convert to

A W e e T e s t

A	65
space	32
W	87
e	101
e	101
space	32
T	84
e	101
s	115
t	116

Figure 3.19 A String of Characters Encoded in ASCII

When the characters are organized into a string to be transmitted and *LF* and/or *CR* code are often put at the end to indicate the end of a line. When stored in a computer an ASCII value of zero is used to end the string.

3.3.2 Parity

Errors often occur when data is transmitted or stored. This is very important when transmitting data in noisy factories, over phone lines, etc. Parity bits can be added to data as a simple check of transmitted data for errors. If the data contains error it can be retransmitted, or ignored.

A parity bit is normally a 9th bit added onto an 8 bit byte. When the data is encoded the number of true bits are counted. The parity bit is then set to indicate if there are an even or odd number of true bits. When the byte is decoded the parity bit is checked to make sure it that there are an even or odd number of data bits true. If the parity bit is not satisfied, then the byte is judged to be in error. There are two types of parity, even or odd. These are both based upon an even or odd number of data bits being true. The odd parity bit is true if there are an odd number of bits on in a binary number. On the other hand the Even parity is set if there are an even number of true bits. This is illustrated in Figure 3.20.

	data bits	parity bit
Odd Parity	10101110	1
	10111000	0
Even Parity	00101010	0
	10111101	1

Figure 3.20 Parity Bits on a Byte

Parity bits are normally suitable for single bytes, but are not reliable for data with a number of bits.

Note: Control systems perform important tasks that can be dangerous in certain circumstances. If an error occurs there could be serious consequences. As a result error detection methods are very important for control system. When error detection occurs the system should either be *robust* enough to recover from the error, or the system should *fail-safe*. If you ignore these design concepts you will eventually cause an accident.

3.3.3 Checksums

Parity bits are suitable for a few bits of data, but checksums are better for larger data transmissions. These are simply an algebraic sum of all of the data transmitted. Before data is transmitted the numeric values of all of the bytes are added. This sum is then transmitted with the data. At the receiving end the data values are summed again, and the total is compared to the checksum. If they match the data is accepted as good. An example of this method is shown in Figure 3.21.

DATA	
	124
	43
	255
	9
	27
	47
CHECKSUM	
	505

Figure 3.21 A Checksum

Checksums are very common in data transmission, but these are also hidden from the average user. If you plan to transmit data to or from a PLC you will need to consider parity and checksum values to verify the data. Small errors in data can have major consequences in received data. Consider an oven temperature transmitted as a binary integer (1023d = 0000 0100 0000 0000b). If a single bit were to be changed, and was not detected the temperature might become (0000 0110 0000 0000b = 1535d) This small change would dramatically change the process.

3.3.4 Gray Code

Parity bits and checksums are for checking data that may have any value. Gray code is used for checking data that must follow a binary sequence. This is common for devices such as angular encoders. The concept is that as the binary number counts up or down, only one bit changes at a time. Thus making it easier to detect erroneous bit changes. An example of a gray code sequence is shown in Figure 3.22. Notice that only one bit changes from one number to the next. If more than a single bit changes between numbers, then an error can be detected.

ASIDE: When the signal level in a wire rises or drops, it induces a magnetic pulse that excites a signal in other nearby lines. This phenomenon is known as *cross-talk*. This signal is often too small to be noticed, but several simultaneous changes, coupled with background noise could result in erroneous values.

decimal	gray code
0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9	1101
10	1111
11	1110
12	1010
13	1011
14	1001
15	1000

Figure 3.22 Gray Code for a Nibble

3.4 Summary

- Binary, octal, decimal and hexadecimal numbers were all discussed.
- 2s compliments allow negative binary numbers.
- BCD numbers encode digits in nibbles.
- ASCII values are numerical equivalents for common alphanumeric characters.
- Gray code, parity bits and checksums can be used for error detection.

3.5 Problems

1. Why are binary, octal and hexadecimal used for computer applications?

2. Is a word is 3 nibbles?

3. What are the specific purpose for Gray code and parity?

4. Convert the following numbers to/from binary

a) from base 10: 54,321 b) from base 2: 110000101101

5. Convert the BCD number below to a decimal number,

0110 0010 0111 1001

6. Convert the following binary number to a BCD number,

0100 1011

7. Convert the following binary number to a Hexadecimal value,

0100 1011

8. Convert the following binary number to a octal,

0100 1011

9. Convert the decimal value below to a binary byte, and then determine the odd parity bit,

97

10. Convert the following from binary to decimal, hexadecimal, BCD and octal.

a) 101101

b) 11011011

c) 10000000001

d) 0010110110101

11. Convert the following from decimal to binary, hexadecimal, BCD and octal.

a)	1	c)	20456
b)	17	d)	-10

12. Convert the following from hexadecimal to binary, decimal, BCD and octal.

a)	1	c)	ABC
b)	17	d)	-A

13. Convert the following from BCD to binary, decimal, hexadecimal and octal.

a)	1001	c)	0011 0110 0001
b)	1001 0011	d)	0000 0101 0111 0100

14. Convert the following from octal to binary, decimal, hexadecimal and BCD.

a)	7	c)	777
b)	17	d)	32634

15.

a) Represent the decimal value thumb wheel input, 3532, as a Binary Coded Decimal (BCD) and a Hexadecimal Value (without using a calculator).
   i) BCD
   ii) Hexadecimal

b) What is the corresponding decimal value of the BCD value, 1001111010011011?

16. Add/subtract/multiply/divide the following numbers.

a) binary 101101101 + 01010101111011
b) hexadecimal 101 + ABC
c) octal 123 + 777
d) binary 110110111 - 0101111
e) hexadecimal ABC - 123
f) octal 777 - 123
g) binary 0101111 - 110110111
h) hexadecimal 123-ABC
i) octal 123 - 777
j) 2s complement bytes 10111011 + 00000011
k) 2s complement bytes 00111011 + 00000011
l) binary 101101101 * 10101
m) octal 123 * 777
n) octal 777 / 123
o) binary 101101101 / 10101
p) hexadecimal ABC / 123

17. Do the following operations with 8 bit bytes, and indicate the condition of the overflow and carry bits.
   a) 10111011 + 00000011
   b) 00111011 + 00000011
   c) 11011011 + 11011111
   d) 110110111 - 01011111
   e) 01101011 + 01111011
   f) 10110110 - 11101110

18. Consider the three BCD numbers listed below.

   1001 0110 0101 0001
   0010 0100 0011 1000
   0100 0011 0101 0001

   a) Convert these numbers to their decimal values.
   b) Convert the decimal values to binary.
   c) Calculate a checksum for all three binary numbers.
   d) What would the even parity bits be for the binary words found in b).

19. Is the 2nd bit set in the hexadecimal value F49?

20. Explain where grey code occurs when creating Karnaugh maps.

21. Convert the decimal number 1000 to a binary number, and then to hexadecimal.

3.6 Problems Solutions

1. base 2, 4, 8, and 16 numbers translate more naturally to the numbers stored in the computer.

2. no, it is four nibbles

3. Both of these are coding schemes designed to increase immunity to noise. A parity bit can be used to check for a changed bit in a byte. Gray code can be used to check for a value error in a stream of continuous values.

4. a) 1101 0100 0011 0001, b) 3117

5. 6279

6. 0111 0101

7. 4B

8. 113

9. 1100001 odd parity bit = 1

10.

binary	101101	11011011	10000000001	0010110110101
BCD	0100 0101	0010 0001 1001	0001 0000 0010 0101	0001 0100 0110 0001
decimal	45	219	1025	1461
hex	2D	5D	401	5B5
octal	55	333	2001	2665

11.

decimal	1	17	20456	-10
BCD	0001	0001 0111	0010 0000 0100 0101 0110	-0001 0000
binary	1	10001	0100 1111 1110 1000	1111 1111 1111 0110
hex	1	11	4FE8	FFF6
octal	1	21	47750	177766

12.

hex	1	17	ABC	-A
BCD	0001	0010 0011	0010 0111 0100 1000	-0001 0000
binary	1	10111	0000 1010 1011 1100	1111 1111 1111 0110
decimal	1	23	2748	-10
octal	1	27	5274	177766

13.

BCD	1001	1001 0011	0011 0110 0001	0000 0101 0111 0100
binary	1001	101 1101	1 0110 1001	10 0011 1110
decimal	9	93	361	0574
hex	9	5D	169	23E
octal	11	135	551	1076

14.

octal	7	17	777	32634
binary	111	1111	1 1111 1111	0011 0101 1001 1100
decimal	7	15	511	13724
hex	7	F	1FF	359C
BCD	0111	0001 0101	0101 0001 0001	0001 0011 0111 0010 0100

15. a) 3532 = 0011 0101 0011 0010 = DCC, b0 the number is not a valid BCD

16.

a) 0001 0110 1110 1000

b) BBD

c) 1122

d) 0000 0001 1000 1000

e) 999

f) 654

g) 1111 1110 0111 1000

h) -999

i) -654

j) 0000 0001 0111 1010

k) 0000 0000 0011 1110

l) 0001 1101 1111 0001

m) 122655

n) 6

o) 0000 0000 0001 0001

p) 9

17.

a) 10111011 + 00000011=1011 1110

b) 00111011 + 00000011=0011 1110

c) 11011011 + 11011111=1011 1010+C+O

d) 110110111 - 01011111=0101 1000+C+O

e) 01101011 + 01111011=1110 0110

f) 10110110 - 11101110=1100 1000

18. a) 9651, 2438, 4351, b) 0010 0101 1011 0011, 0000 1001 1000 0110, 0001 0000 1111 1111, c) 16440, d) 1, 0, 0

19. The binary value is 1111 0100 1001, so the second bit is 0

20. when selecting the sequence of bit changes for Karnaugh maps, only one bit is changed at a time. This is the same method used for grey code number sequences. By using the code the bits in the map are naturally grouped.

21.

$$1000_{10} = 1111101000_2 = 3e8_{16}$$

3.7 Challenge Problems

1.

4. TRANSFORMS

***** This contains additions and sections by Dr. Andrew Sterian.

Topics:

-

Objectives:

-

4.1 Laplace Transforms

• The Laplace transform allows us to reverse time. And, as you recall from before the inverse of time is frequency. Because we are normally concerned with response, the Laplace transform is much more useful in system analysis.

• The basic Laplace transform equations is shown below,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where,

$$f(t) = \text{the function in terms of time t}$$

$$F(s) = \text{the function in terms of the Laplace s}$$

4.1.1 Laplace Transform Tables

• Basic Laplace Transforms for operational transformations are given below,

TIME DOMAIN	FREQUENCY DOMAIN
$Kf(t)$	$Kf(s)$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sf(s) - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 f(s) - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n f(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^n f(0^-)}{dt^n}$
$\int_0^t f(t) dt$	$\frac{f(s)}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as} f(s)$
$e^{-at} f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty f(u) du$

• A set of useful functional Laplace transforms are given below,

TIME DOMAIN	FREQUENCY DOMAIN
A	$\frac{A}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Ae^{-at}	$\frac{A}{s-a}$
Ate^{-at}	$\frac{A}{(s-a)^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$

- Laplace transforms can be used to solve differential equations.

4.2 z-Transforms

- For a discrete-time signal $x[n]$, the two-sided z-transform is defined by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$.

  The one-sided z-transform is defined by $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$. In both cases, the z-transform is a polynomial in the complex variable z.

- The inverse z-transform is obtained by contour integration in the complex plane $x[n] = \frac{1}{j2\pi}\oint X(z)z^{n-1}dz$. This is usually avoided by partial fraction inversion techniques, similar to the Laplace transform.

- Along with a z-transform we associate its region of convergence (or ROC). These are the values of z for which $X(z)$ is bounded (i.e., of finite magnitude).

• Some common z-transforms are shown below.

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$(-a^n)u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(-na^n)u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u[n]$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$a^n \sin(\omega_0 n) u[n]$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$\dfrac{n!}{k!(n-k)!}u[n]$	$\dfrac{z^{-k}}{(1-z^{-1})^{k+1}}$	$\lvert z\rvert > 1$

• The z-transform also has various properties that are useful. The table below lists properties for the two-sided z-transform. The one-sided z-transform properties can be derived from the ones below by considering the signal $x[n]u[n]$ instead of simply $x[n]$.

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	$r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$\alpha x_1[n] + \beta x_2[n]$	$\alpha X_1(z) + \beta X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x[n-k]$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
z-Domain Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time Reversal	$x[-n]$	$X(z^{-1})$	$\dfrac{1}{r_1} < \lvert z\rvert < \dfrac{1}{r_2}$
z-Domain Differentiation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z \rvert < r_{1u}r_{2u}$
Initial value theorem	$x[n]$ causal	$x[0] = \lim_{z \to \infty} X(z)$	

4.3 Fourier Series

- These series describe functions by their frequency spectrum content. For example a square wave can be approximated with a sum of a series of sine waves with varying magnitudes.

- The basic definition of the Fourier series is given below.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

4.4 Problems

8a. Find y(t).

$$\frac{y(s)}{x(s)} = \frac{s^2+4s}{s^2+6s+9} \qquad x(t) = 5$$

4.5 Challenge Problems

5. GEOMETRY

Topics:

-

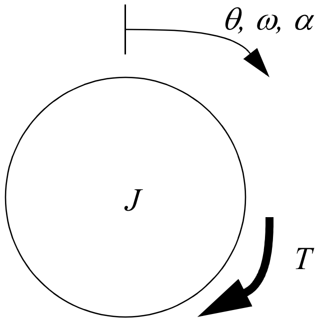
Objectives:

-

5.1 Introduction

5.1.1 Inertia

When unbalanced torques are applied to a mass it will begin to accelerate, in rotation. The sum of applied torques is equal to the inertia forces shown in Figure 1.23.



$$\sum T = J_M \alpha \qquad (6)$$

$$J_M = I_{xx} + I_{yy} \qquad (7)$$

$$I_{xx} = \int y^2 dM \qquad (8)$$

$$I_{yy} = \int x^2 dM \qquad (9)$$

Note: The 'mass' moment of inertia will be used when dealing with acceleration of a mass. Later we will use the 'area' moment of inertia for torsional springs.

Figure 1.23 Summing moments and angular inertia

The mass moment of inertia determines the resistance to acceleration. This can be calculated using integration, or found in tables. When dealing with rotational acceleration it is important to use the mass moment of inertia, not the area moment of inertia.

The center of rotation for free body rotation will be the centroid. Moment of inertia values are typically calculated about the centroid. If the object is constrained to rotate about some point, other than the centroid, the moment of inertia value must be recalculated. The parallel axis theorem provides the method to shift a moment of inertia from a centroid to an arbitrary center of rotation, as shown in Figure 1.24.

$$J_M = \tilde{J}_M + Mr^2$$

where,

J_M = *mass moment about the new point*

$\tilde{J}_M$ = *mass moment about the center of mass*

M = *mass of the object*

r = *distance from the centroid to the new point*

Figure 1.24 Parallel axis theorem for shifting a mass moment of inertia

$$J_A = \tilde{J}_A + Ar^2$$

where,

J_A = *area moment about the new point*

$\tilde{J}_A$ = *area moment about the centroid*

A = *mass of the object*

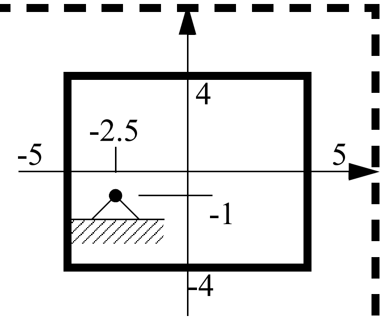
r = *distance from the centroid to the new point*

Figure 1.25 Parallel axis theorem for shifting a area moment of inertia

Aside: If forces do not pass through the center of an object, it will rotate. If the object is made of a homogeneous material, the area and volume centroids can be used as the center. If the object is made of different materials then the center of mass should be used for the center. If the gravity varies over the length of the (very long) object then the center of gravity should be used.

An example of calculating a mass moment of inertia is shown in Figure 1.26. In this problem the density of the material is calculated for use in the integrals. The integrals are then developed using slices for the integration element dM. The integrals for the moments about the x and y axes, are then added to give the polar moment of inertia. This is then shifted from the centroid to the new axis using the parallel axis theorem.

The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia.



First find the density and calculate the moments of inertia about the centroid.

$$\rho = \frac{10Kg}{2(5m)2(4m)} = 0.125Kgm^{-2}$$

$$I_{xx} = \int_{-4}^{4} y^2 dM = \int_{-4}^{4} y^2 \rho 2(5m) dy = 1.25Kgm^{-1}\frac{y^3}{3}\bigg|_{-4}^{4}$$

$$\therefore \quad = 1.25Kgm^{-1}\left(\frac{(4m)^3}{3} - \frac{(-4m)^3}{3}\right) = 53.33Kgm^2$$

$$I_{yy} = \int_{-5}^{5} x^2 dM = \int_{-5}^{5} x^2 \rho 2(4m) dx = 1Kgm^{-1}\frac{x^3}{3}\bigg|_{-5}^{5}$$

$$\therefore \quad = 1Kgm^{-1}\left(\frac{(5m)^3}{3} - \frac{(-5m)^3}{3}\right) = 83.33Kgm^2$$

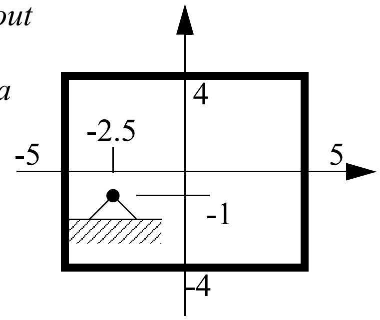
$$J_M = I_{xx} + I_{yy} = 53.33Kgm^2 + 83.33Kgm^2 = 136.67Kgm^2$$

The centroid can now be shifted to the center of rotation using the parallel axis theorem.

$$J_M = \tilde{J}_M + Mr^2 = 136.67Kgm^2 + (10Kg)((-2.5m)^2 + (-1m)^2) = 209.2Kgm^2$$

Figure 1.26 Mass moment of inertia example

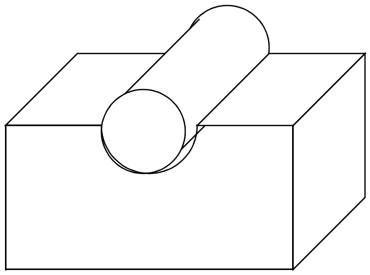
The rectangular shape to the right is constrained to rotate about point A. The total mass of the object is 10kg. The given dimensions are in meters. Find the mass moment of inertia WITHOUT using the parallel axis theorem.



ans.

$$I_{M_x} = 66.33 Kgm^2$$
$$I_{M_y} = 145.8 Kgm^2$$
$$J_M = 209.2 Kgm^2$$

Figure 1.27 Drill problem: Mass moment of inertia calculation



The 20cm diameter 10 kg cylinder to the left is sitting in a depression that is effectively frictionless. If a torque of 10 Nm is applied for 5 seconds, what will the angular velocity be?

ans. $\theta(5s) = 312.5 rad$

$\omega(5s) = 125\frac{rad}{s}$

Figure 1.28 Drill problem: Find the velocity of the rotating shaft

• A set of the basic 2D and 3D geometric primitives are given, and the notation used is described below,

A = contained area

P = perimeter distance

V = contained volume

S = surface area

x, y, z = center of mass

$\bar{x}, \bar{y}, \bar{z}$ = centroid

I_x, I_y, I_z = moment of inertia of area (or second moment of inertia)

AREA PROPERTIES:

$$I_x = \int_A y^2 dA = \text{the moment of inertia about the y-axis}$$

$$I_y = \int_A x^2 dA = \text{the moment of inertia about the x-axis}$$

$$I_{xy} = \int_A xy dA = \text{the product of inertia}$$

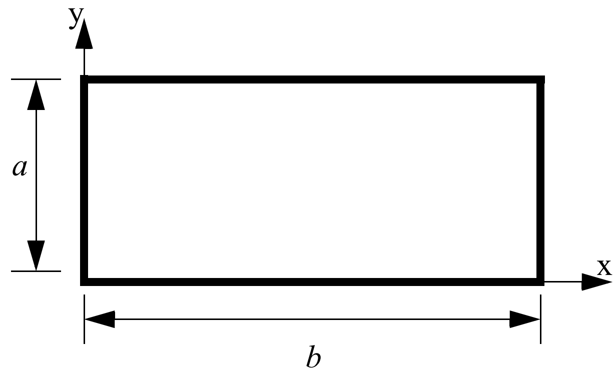
$$J_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y = \text{The polar moment of inertia}$$

$$\bar{x} = \frac{\int_A x dA}{\int_A dA} = \text{centroid location along the x-axis}$$

$$\bar{y} = \frac{\int_A y dA}{\int_A dA} = \text{centroid location along the y-axis}$$

Rectangle/Square:

$$A = ab$$
$$P = 2a + 2b$$



Centroid:

$$\bar{x} = \frac{b}{2}$$
$$\bar{y} = \frac{a}{2}$$

Moment of Inertia (about centroid axes):

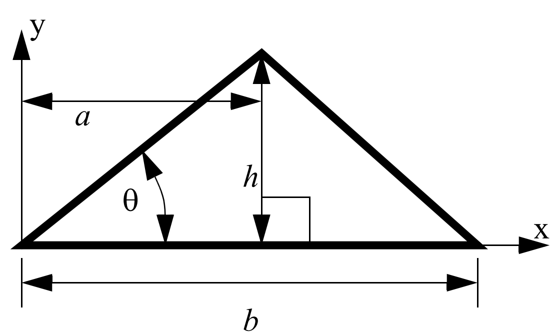
$$\overline{I_x} = \frac{ba^3}{12}$$
$$\overline{I_y} = \frac{b^3 a}{12}$$
$$\overline{I_{xy}} = 0$$

Moment of Inertia (about origin axes):

$$I_x = \frac{ba^3}{3}$$
$$I_y = \frac{b^3 a}{3}$$
$$I_{xy} = \frac{b^2 a^2}{4}$$

Triangle:

$$A = \frac{bh}{2}$$
$$P =$$



Centroid:

$$\bar{x} = \frac{a+b}{3}$$
$$\bar{y} = \frac{h}{3}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = \frac{bh^3}{36}$$
$$\overline{I_y} = \frac{bh}{36}(a^2 + b^2 - ab)$$
$$\overline{I_{xy}} = \frac{bh^2}{72}(2a - b)$$

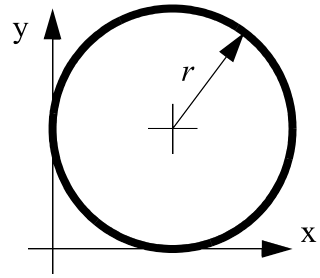
Moment of Inertia (about origin axes):

$$I_x = \frac{bh^3}{12}$$
$$I_y = \frac{bh}{12}(a^2 + b^2 - ab)$$
$$I_{xy} = \frac{bh^2}{24}(2a - b)$$

Circle:

$$A = \pi r^2$$

$$P = 2\pi r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = r$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = \frac{\pi r^4}{4}$$

$$\overline{I_y} = \frac{\pi r^4}{4}$$

$$\overline{I_{xy}} = 0$$

Moment of Inertia (about origin axes):

$$I_x =$$

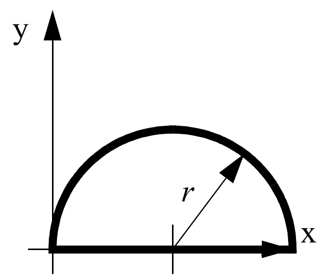
$$I_y =$$

$$I_{xy} =$$

Half Circle:

$$A = \frac{\pi r^2}{2}$$

$$P = \pi r + 2r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$$

$$\overline{I_y} = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Moment of Inertia (about origin axes):

$$I_x = \frac{\pi r^4}{8}$$

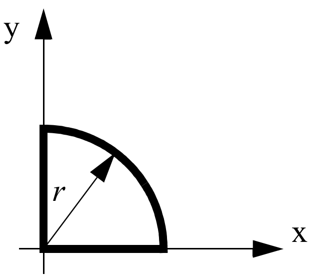
$$I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Quarter Circle:

$$A = \frac{\pi r^2}{4}$$

$$P = \frac{\pi r}{2} + 2r$$



Centroid:

$$\bar{x} = \frac{4r}{3\pi}$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = 0.05488 r^4$$

$$\overline{I_y} = 0.05488 r^4$$

$$\overline{I_{xy}} = -0.01647 r^4$$

Moment of Inertia (about origin axes):

$$I_x = \frac{\pi r^4}{16}$$

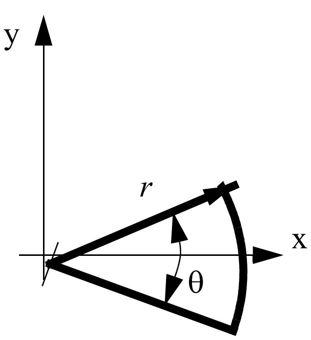
$$I_y = \frac{\pi r^4}{16}$$

$$I_{xy} = \frac{r^4}{8}$$

Circular Arc:

$$A = \frac{\theta r^2}{2}$$

$$P = \theta r + 2r$$



Centroid:

$$\bar{x} = \frac{2r\sin\frac{\theta}{2}}{3\theta}$$

$$\bar{y} = 0$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} =$$

$$\overline{I_y} =$$

$$\overline{I_{xy}} =$$

Moment of Inertia (about origin axes):

$$I_x = \frac{r^4}{8}(\theta - \sin\theta)$$

$$I_y = \frac{r^4}{8}(\theta + \sin\theta)$$

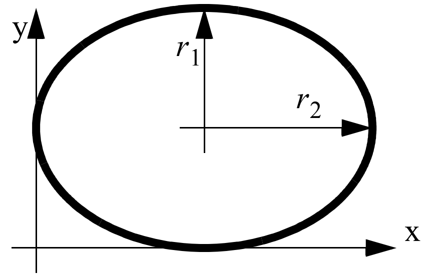
$$I_{xy} = 0$$

Ellipse:

$$A = \pi r_1 r_2$$

$$P = 4r_1\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{\sqrt{r_1^2+r_2^2}}{a}(\sin\theta)^2}\,d\theta$$

$$P \approx 2\pi\sqrt{\frac{r_1^2+r_2^2}{2}}$$



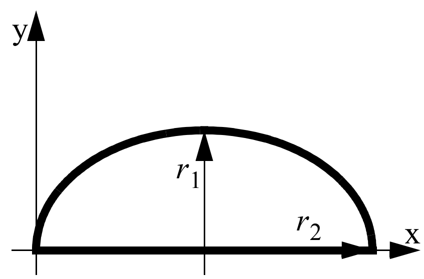
Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\overline{I_x} = \dfrac{\pi r_1^3 r_2}{4}$	$I_x =$
$\bar{y} = r_1$	$\overline{I_y} = \dfrac{\pi r_1 r_2^3}{4}$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Ellipse:

$$A = \frac{\pi r_1 r_2}{2}$$

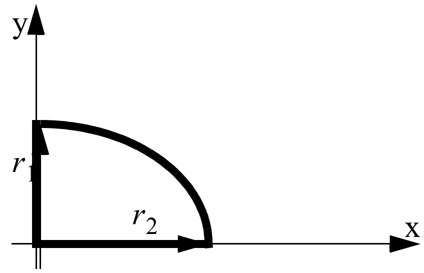
$$P = 2r_1\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{\sqrt{r_1^2+r_2^2}}{a}(\sin\theta)^2}\,d\theta + 2r_2$$

$$P \approx \pi\sqrt{\frac{r_1^2+r_2^2}{2}} + 2r_2$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\overline{I_x} = 0.05488 r_2 r_1^3$	$I_x = \dfrac{\pi r_2 r_1^3}{16}$
$\bar{y} = \dfrac{4r_1}{3\pi}$	$\overline{I_y} = 0.05488 r_2^3 r_1$	$I_y = \dfrac{\pi r_2^3 r_1}{16}$
	$\overline{I_{xy}} = -0.01647 r_1^2 r_2^2$	$I_{xy} = \dfrac{r_1^2 r_2^2}{8}$

Quarter Ellipse:



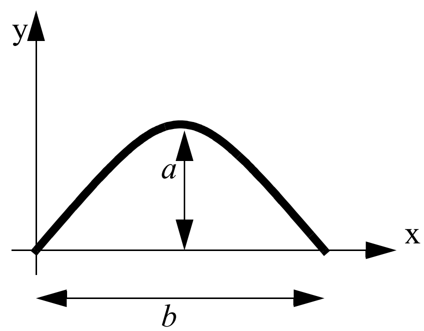
$$A = \frac{\pi r_1 r_2}{4}$$

$$P = r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2}\, d\theta + 2r_2$$

$$P \approx \frac{\pi}{2}\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{4r_2}{3\pi}$	$\overline{I_x} =$	$I_x = \pi r_2 r_1^3$
$\bar{y} = \frac{4r_1}{3\pi}$	$\overline{I_y} =$	$I_y = \pi r_2^3 r_1$
	$\overline{I_{xy}} =$	$I_{xy} = \frac{r_2^2 r_1^2}{8}$

Parabola:



$$A = \frac{2}{3}ab$$

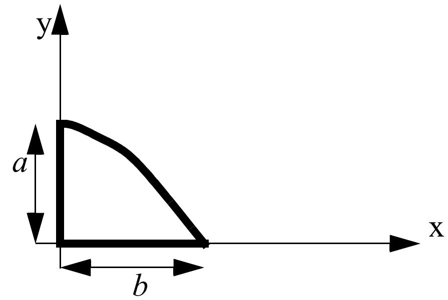
$$P = \frac{\sqrt{b^2 + 16a^2}}{2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{b}{2}$	$\overline{I_x} =$	$I_x =$
$\bar{y} = \frac{2a}{5}$	$\overline{I_y} =$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Parabola:

$$A = \frac{ab}{3}$$

$$P = \frac{\sqrt{b^2 + 16a^2}}{4} + \frac{b^2}{16a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{3b}{8}$	$\overline{I_x} = \frac{8ba^3}{175}$	$I_x = \frac{2ba^3}{7}$
$\bar{y} = \frac{2a}{5}$	$\overline{I_y} = \frac{19b^3a}{480}$	$I_y = \frac{2b^3a}{15}$
	$\overline{I_{xy}} = \frac{b^2a^2}{60}$	$I_{xy} = \frac{b^2a^2}{6}$

- A general class of geometries are conics. This for is shown below, and can be used to represent many of the simple shapes represented by a polynomial.

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Conditions

$A = B = C = 0$	straight line
$B = 0, A = C$	circle
$B^2 - AC < 0$	ellipse
$B^2 - AC = 0$	parabola
$B^2 - AC > 0$	hyperbola

VOLUME PROPERTIES:

$$I_x = \int_V r_x^{\,2}\, dV = \text{ the moment of inertia about the x-axis}$$

$$I_y = \int_V r_y^{\,2}\, dV = \text{ the moment of inertia about the y-axis}$$

$$I_z = \int_V r_z^{\,2}\, dV = \text{ the moment of inertia about the z-axis}$$

$$\bar{x} = \frac{\int_V x\,dV}{\int_V dV} = \text{ centroid location along the x-axis}$$

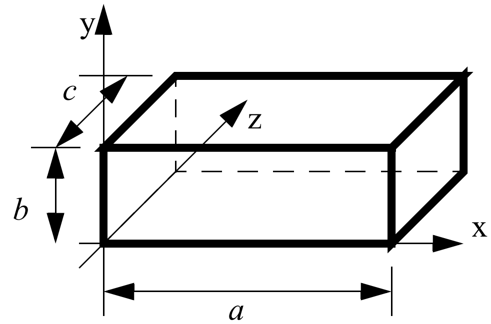
$$\bar{y} = \frac{\int_V y\,dV}{\int_V dV} = \text{ centroid location along the y-axis}$$

$$\bar{z} = \frac{\int_V z\,dV}{\int_V dV} = \text{ centroid location along the z-axis}$$

Parallelepiped (box):

$V = abc$

$S = 2(ab + ac + bc)$

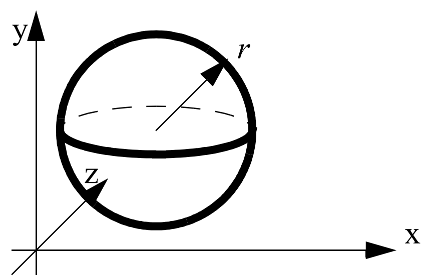


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{a}{2}$	$\bar{I}_x = \frac{M(a^2 + b^2)}{12}$	$I_x =$
$\bar{y} = \frac{b}{2}$	$\bar{I}_y = \frac{M(a^2 + c^2)}{12}$	$I_y =$
$\bar{z} = \frac{c}{2}$	$\bar{I}_z = \frac{M(b^2 + a^2)}{12}$	$I_z =$

Sphere:

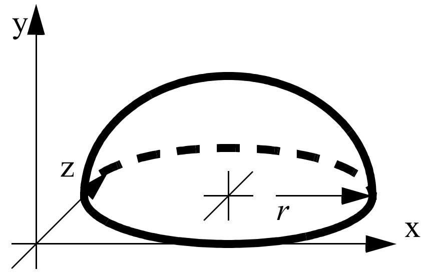
$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = \frac{2Mr^2}{5}$	$I_x =$
$\bar{y} = r$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = \frac{2Mr^2}{5}$	$I_z =$

Hemisphere:

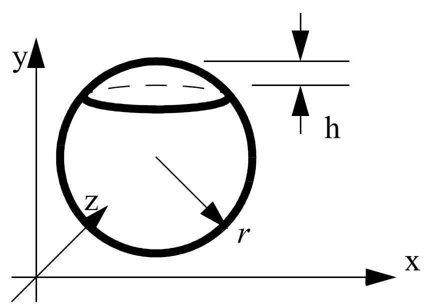


$$V = \frac{2}{3}\pi r^3$$

$$S =$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = \frac{83}{320}Mr^2$	$I_x =$
$\bar{y} = \frac{3r}{8}$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = \frac{83}{320}Mr^2$	$I_z =$

Cap of a Sphere:

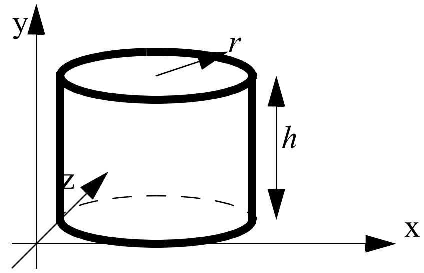


$$V = \frac{1}{3}\pi h^2(3r - h)$$

$$S = 2\pi r h$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x =$	$I_x =$
$\bar{y} =$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r$	$\bar{I}_z =$	$I_z =$

Cylinder:

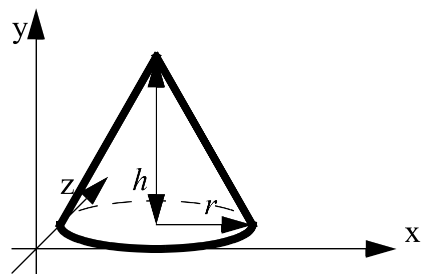


$$V = h\pi r^2$$

$$S = 2\pi r h + 2\pi r^2$$

Centroid:	Moment of Inertia (about centroid axis):	Moment of Inertia (about origin axis):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_x = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$
$\bar{y} = \frac{h}{2}$	$\bar{I}_y = \frac{Mr^2}{2}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_z = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$

Cone:



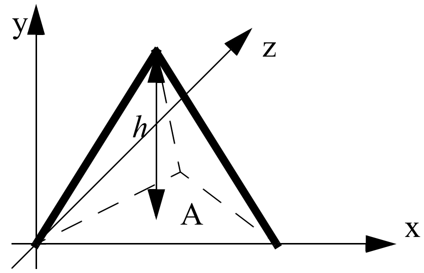
$$V = \frac{1}{3}\pi r^2 h$$

$$S = \pi r\sqrt{r^2 + h^2}$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y = \frac{3Mr^2}{10}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_z =$

Tetrahedron:

$$V = \frac{1}{3}Ah$$

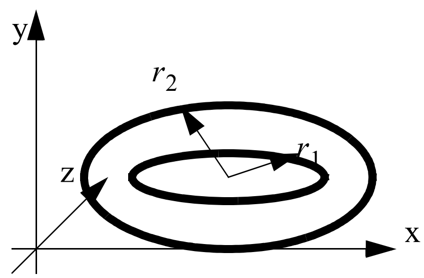


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} =$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y =$	$I_y =$
$\bar{z} =$	$\bar{I}_z =$	$I_z =$

Torus:

$$V = \frac{1}{4}\pi^2(r_1 + r_2)(r_2 - r_1)^2$$

$$S = \pi^2(r_2^2 - r_1^2)$$

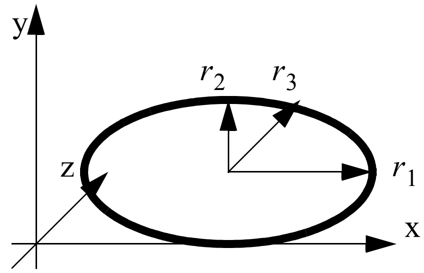


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \left(\frac{r_2 - r_1}{2}\right)$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_2$	$\bar{I}_z =$	$I_z =$

Ellipsoid:

$$V = \frac{4}{3}\pi r_1 r_2 r_3$$

$$S =$$

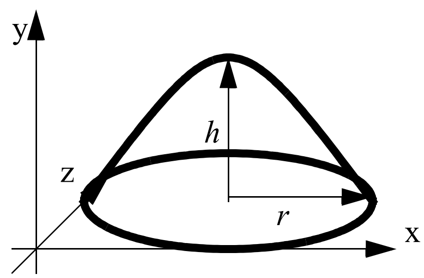


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_1$	$\overline{I_x} =$	$I_x =$
$\bar{y} = r_2$	$\overline{I_y} =$	$I_y =$
$\bar{z} = r_3$	$\overline{I_z} =$	$I_z =$

Paraboloid:

$$V = \frac{1}{2}\pi r^2 h$$

$$S =$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\overline{I_x} =$	$I_x =$
$\bar{y} =$	$\overline{I_y} =$	$I_y =$
$\bar{z} = r$	$\overline{I_z} =$	$I_z =$

5.2 Problems

5.2.1 Challenge Problems

6. FINANCIAL

Topics:

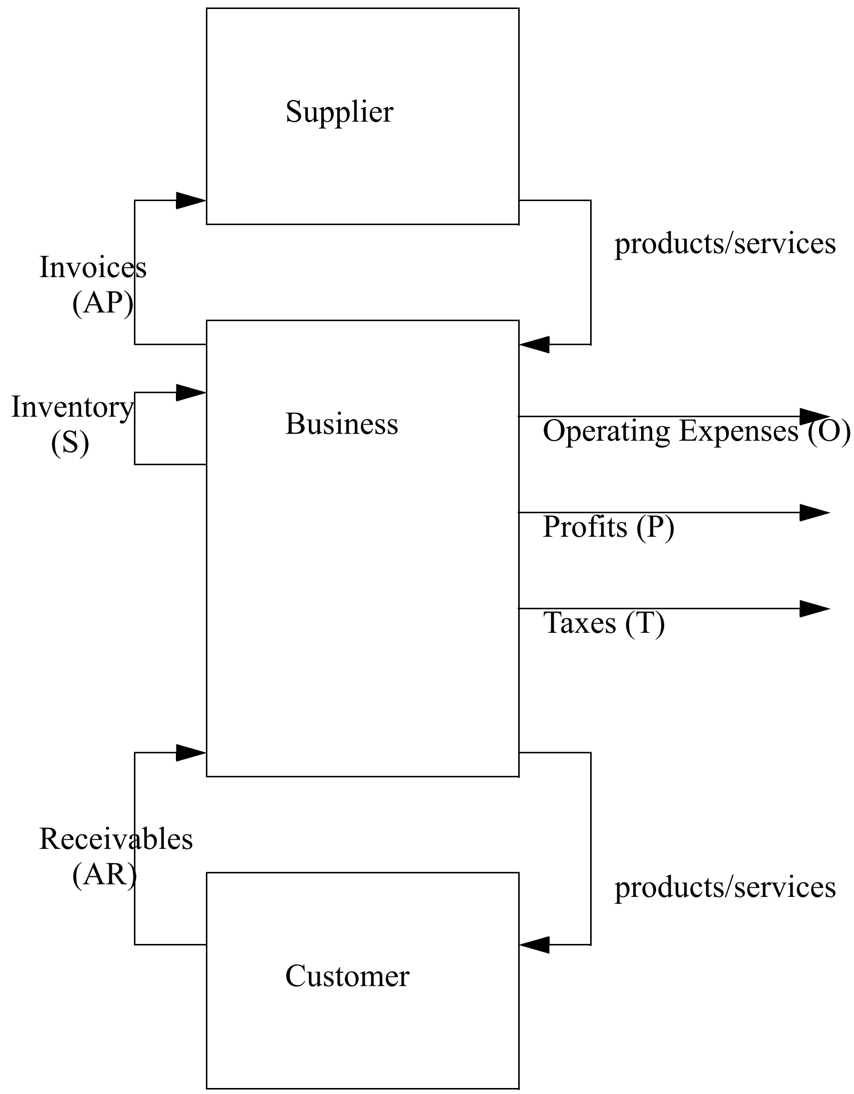
•

Objectives:

•

6.1 Introduction

• The primary object of a public corporation is to generate the maximum amount of profit possible. A simple model is shown below.



$$T = 0.5P \qquad \text{(assumes 50\% tax on profit)}$$

$$\sum = AR - AP - O - P - T = 0$$

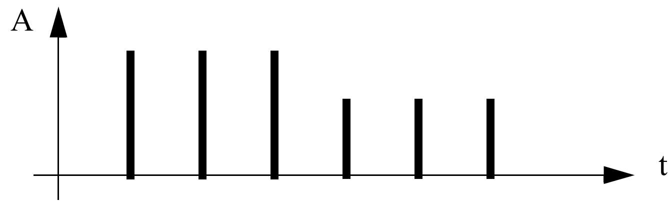
$$\therefore AR - AP - O - P - 0.5P = 0$$

$$\therefore P = \frac{AR - AP - O}{1.5}$$

• Operating expenses include a number of factors such as,
- Depreciation on capital equipment

- Wages/salaries

• When cash flow occurs over a number of years it is often shown with cash flow diagrams.



• When considering the economic value of a decision, one method is the payback period.

$$N = \frac{C_I}{S_A}$$

where,

C_I = initial investment ($)
S_A = savings per year ($/yr)
N = payback period (years)

• Simple estimates for the initial investment and yearly savings are,

$$C_I = C_E - I_S$$

where,

C_E = cost of new equipment
I_S = revenue from sale of old equipment (salvage)

$$S_A = (L_0 H_0 - L_1 H_1) + (M_0 - M_1)$$

where,

L_0, L_1 = labor rate before and after
H_0, H_1 = labor hours before and after
M_0, M_1 = maintenance costs before and after

• There are clearly more factors than can be considered, including,
  - changes in material use
  - opportunity cost
  - setup times
  - change in inventory size
  - material handling change

• The simple models ignore the conversion between present value and future value. (ie, money now is worth more than the same amount of money later)

$$PW = C_0 + \sum[(R_{A_j} - C_{A_j})(P/F, i, j)]$$

$$(P/F, i, j) = \frac{1}{(1+i)^j} \qquad (P/A, i, n) = \sum(P/F, i, j) = \frac{(1+i)^n - 1}{i(1+i)^n}$$

where,

$PW =$ present worth of the money (in todays dollars)

$R_{A_j} =$ Annual revenues (income) for year j

$C_{A_j} =$ Annual costs (expenses) for year j

$j =$ j years in the future

$i =$ interest rate (fractional)

$n =$ number of years for consideration

• The future value of money can be evaluated using,

$$F = P(F/P, i, n) \qquad (F/P, i, n) = (1+i)^n$$

where,

$FW =$ future worth of the money

e.g.,

$i = 5\% \qquad P = 1000 \qquad n = 3$

$$F = P(F/P, i, n) = 1000(F/P, 5\%, 3) = 1000(1 + 0.05)^3 =$$

• Quite often a Rate of Return (ROR) will be specified by management. This is used in place of interest rates, and can include a companies value for the money. This will always be higher than the typical prime interest rate.

• So far we haven't considered the effects of taxes. Basically corporate taxes are applied to profits. Therefore we attempt to distribute expenses evenly across the life of a project (even though the majority of the money has been spent in the first year). This distribution is known as depreciation.

$$A = B - T = B - (tax_{rate}C) = B(1 - tax_{rate}) + Dtax_{rate}$$

where,

A = after tax cash ($/yr)

B = before tax cash ($/yr)

D = depreciation of equipment ($/yr)

tax_{rate} = the corporate tax rate

• Methods for depreciation are specified in the tax laws. One method is straight line depreciation.

$$D = \frac{C_E - I_S}{n}$$

where,

D = The annual depreciation

• Another methods is the accelerated cost recovery (ACRS) method that is based on US tax law. A similar version is the Modified ACRS (MACRS) system.

$$D_j = C \times f_j$$

where

D_j = The annual depreciation for year j

f_j = the depreciation factor for year j

recovery % for a given depreciation period

year	3 yrs	5 yrs	7 yrs	10 yrs
1	33.3	20.0	14.3	10.0
2	44.5	32.0	24.5	18.0
3	14.8	19.2	17.5	14.4
4	7.4	11.5	12.5	11.5
5		11.5	8.8	9.2
6		5.8	8.9	7.4
7			8.9	6.6
8			4.5	6.6
9				6.5
10				6.5
11				3.3

(copied from Lindberg FE Review Manual)

• The 'book value' for an item is calcuated as,

$$BV = C - \sum D_j$$

• Rate of Return (ROR) Analysis -

• Present worth analysis shifts all of the future costs and incomes into a present day sum.

• Benefit Cost Analysis for chosing between alternatives, All costs must be converted to the present values.

$$\frac{B_2 - B_1}{C_2 - C_1} > 1 \qquad \text{Choice 2 is better than 1}$$

• Break Even Analysis

$$C - PBP \times NAP$$

where,

$$PBP = \text{Payback period}$$
$$NAP = \text{Net annual profit}$$

• Return On Investment,

$$ROI = \frac{B - C}{C} 100\%$$

• Consider an assembly line that is currently in use, and the system proposed to replace it. The product line is expected to last 5 years, and then be sold off. The corporate tax rate is 50% and the company policy is to require a 17% rate of return. Should we keep the old line, or install the new one?

Current Manual Line:
- used 2000 hrs/yr with 10 workers at $20/hr each
- maintenance is $20,000/yr
- the current equipment is worth $20,000 used

Proposed Line:
- the equipment will cost $100,000 and the expected salvage value at the end of the project is $10,000
- 2 workers are required for 1000 hours year at $40/hr each
- yearly maintenance will be $40,000

6.2 Problems

1. If $100,000 were borrowed for 3 years at a %10 interest rate, how much would be due at the end of the loan. (ans. $133,100)

2. If $100,000 were borrowed for 3 years at a %10 interest rate, how much would be due at the end of the loan if $20,000 were repaid each year. (ans. $66,900)

3. A machine was purchased for $100,000 and generates $20,000 per year income. How many years would be required to break even if the company charged a 10% internal interest rate. (ans. 7.27 yrs)

4. If a machine is purchased for $100,000 and the company charges %10 for the use of money, what annual return is required for the machine to break even in 3 years. (ans. $40,211.49)

5. A machine costs $100,000 and will be sold for salvage value in 3 years for $30,000. What is the Equivalent Uniform Annual Cost for the machine assuming the company uses an interest rate of 10%? (ans. $31,148.69)

6. A machine costs $100,000 and will be sold for salvage value in 3 years for $30,000. The alternative is to lease a machine for $40,000 per year. If the company uses an interest rate of 10% which option should be chosen? (ans. purchasing is a better option)

7. An existing manual production line costs $100,000 to operate per year. A new piece of automated equipment is being considered to replace the manual production line. The new equipment costs $150,000 and requires $30,000 per year to operate. The decision to purchase the new machine will be based upon a 3 year period with a 25% interest rate. Compare the present value of the two options. (ans. PVmanual $195,200, PVnew $208,560)

8. Write a general computer program to solve the following project costing problem. Test the program using the numbers provided. The program should accept the initial cost of equipment (C), an annual maintenance cost (M), an annual income (R), a salvage value (S), and an interest rate (I). The program should then calculate a present worth and the ROI.

Test values:

$$C = 100,000$$
$$M = 20,000$$
$$R = 150,000$$
$$S = 10,000$$
$$I = 10\%$$
$$L = 3yrs$$

9. Write a program that determines the ROI for a project given the project length, initial cost, salvage value, and projected income. To test the program assume that the project lasts for 36 months. The company standard interest rate is 18%. The equipment will cost $100,000 new and have a slavage value of $10,000. The annual income will be $50,000.

6.3 Challenge Problems

1. Consider Moore's law that predicts that every 18 months the basic speed and capacity of semiconductors will double. We have a customer that wants......

7. OPTIMIZATION

Topics:

-

Objectives:

-

7.1 Introduction

• This is normally used when there is no clearly optimal solution to a problem.

• The basic procedure is,

1. Idenitfy the major variables in a decision
2. Model the system or decisions to be made with equations
3. Assign a cost (objective) function for the system
4. Select an optimization method and search
5. Analyze the results and search again if necessary

7.2 System Modeling and Variable Identification

• Typical decision variables in systems include,

- mass
- volume
- power consumption
- component cost
- factor of safety
- signal-to-noise ratio

- transmission rates
- etc.

• A model of the system should relate the decisions to be made mathematically. These relationships oftem include performance measures,
    - cutting speeds
    - cycle times
    - power
    - capacity
    - flow rates
    - etc.

7.3 Cost Functions and Constraints

• Cost functions quantify things that we want to minimize, or maximize. These should be aligned with system objectives. Typical cost functions include,

- money
- time
- some combination of factors

• Systems also have constraints that limit available solutions

• Expressed as a function of variables that provides a value

• Consider the example of building a fenced pasture. In this case when the area becomes too large, there is a reduced value. We want to maximize the value of V.

$$C_{fence} = 20\left(\frac{\$}{m}\right)(2w + 2d)$$

where,

C_{fence} = cost to construct the fence ($)

w = width of the pasture (m)

d = depth of the pasture (m)

$$C_{land} = 0.35\left(\frac{\$}{m^2}\right)$$

where,

C_{land} = cost for the land

$$R = 0.05\left(\frac{\$}{m^2 yr}\right)wd \qquad wd < 300m^2$$

$$R = 0.01\left(\frac{\$}{m^2 yr}\right)(wd - 300m^2) + 60\$ \qquad wd \ge 300m^2$$

where,

R = revenue generated by pasture land

$$V = R - C_{fence} - C_{land}$$

where,

V = total value

Figure 1.29 Example cost function for building a fence around a pasture

• The cost function can be written as..

```
double cost(double w, double d){
        double value;
        double cfence, cland, R;

        Cfence = 40*(w + d);
        Cland = 0.05*w*d;
        if (w*d < 300){
                        R = 0.05 * w * d;
        } else {
                        R = 0.01 * (w * d - 300) + 60;
        }
        value = R - Cfence - Cland

        return value;
}
```

Figure 1.30 A subroutine for cost function calculation

• Constraints are boundaries that cannot be crossed.

• Example of constraints, the pasture cannot be larger than one 1600m be 1600m beacuse of the constraints of an existing road system.

$$w \le 1600m$$

$$d \le 1600m$$

Figure 1.31 Example constraint functions for a pasture

• The cost function can be written as..

```
double cost(double w, double d){
        double value;
        double cfence, cland, R;

        Cfence = 40*(w + d);
        Cland = 0.05*w*d;
        if (w*d < 300){
                        R = 0.05 * w * d;
        } else {
                        R = 0.01 * (w * d - 300) + 60;
        }
        value = R - Cfence - Cland

        if(w > 1600) value = 1000000;
        if(d > 1600) value = 1000000;

        return value;
}
```

Figure 1.32 A subroutine for cost function calculation

• Slack variables allow constraints to be considered as part of the cost function. Helps with a system with many local minimum.

$$C_{penalty} = \left(\frac{w}{1600m}\right)^4 + \left(\frac{d}{1600m}\right)^4$$

$$V = R - C_{fence} - C_{land} - C_{penalty}$$

Figure 1.33 Example of slack variables for including constraints

• The cost function can be written as..

```
double cost(double w, double d){
        double value;
        double cfence, cland, R;
        double slack;

        Cfence = 40*(w + d);
        Cland = 0.05*w*d;
        if (w*d < 300){
                        R = 0.05 * w * d;
        } else {
                        R = 0.01 * (w * d - 300) + 60;
        }
        // slack variable
        slack = pow((w/1600), 4) + pow((d/1600), 4);
        value = R - Cfence - Cland - slack;

        return value;
}
```

Figure 1.34 A subroutine for cost function calculation

7.4 Single Variable Searches

• For simple single value problems use derivatives and find the maxima/minima.

7.4.1 Example Problem

• As with most engineering problems we want to get the highest return, with the minimum investment. In this case we want to minimize costs, while increasing cutting speeds.

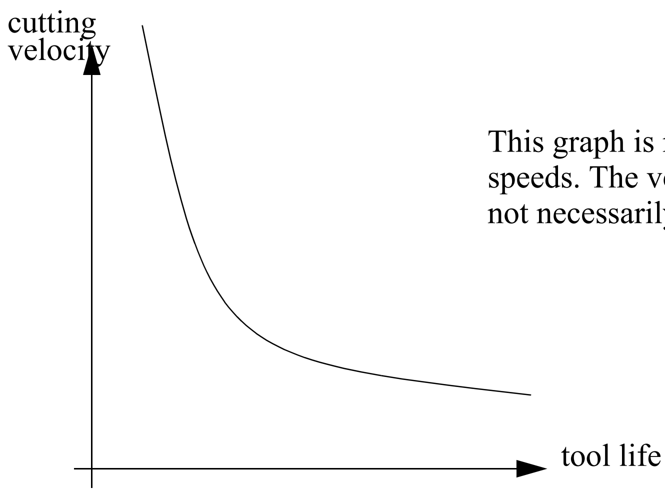
• EFFICIENCY will be the key term - it suggests that good quality parts are produced at reasonable cost.

• Cost is a primarily affected by,
  - tool life
  - power consumed

• The production throughput is primarily affected by,
  - accuracy including dimensions and surface finish
  - mrr (metal removal rate)

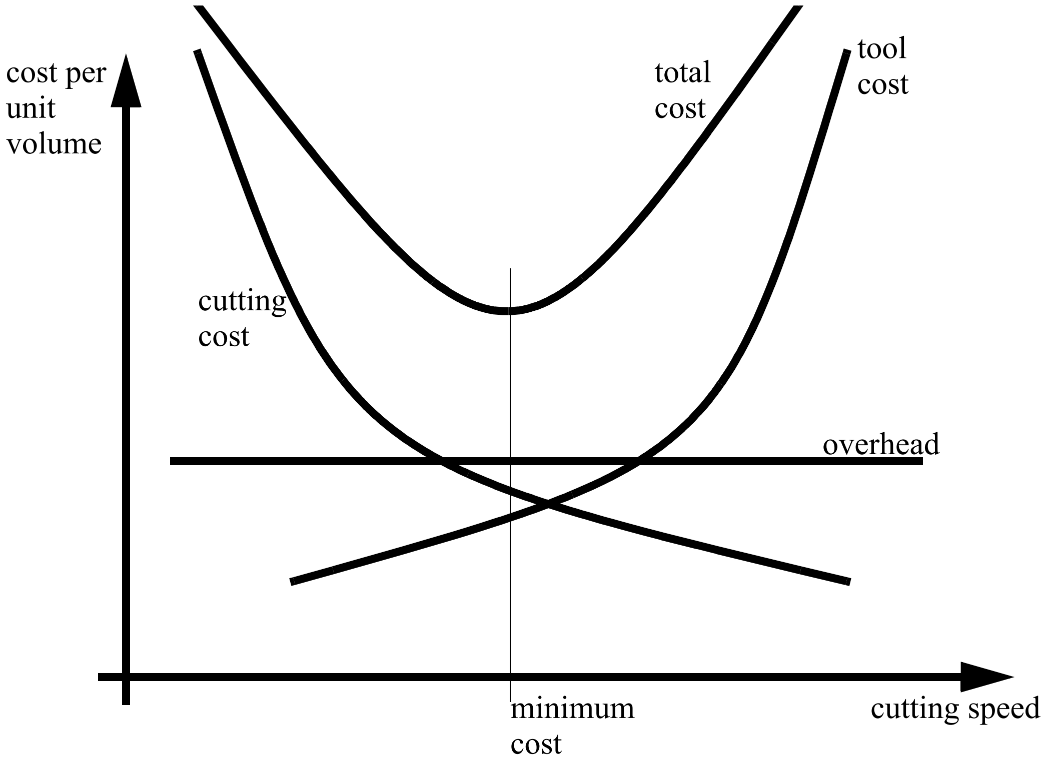
• The factors that can be modified to optimize the process are,
- cutting velocity (biggest effect)
- feed and depth
- work material
- tool material
- tool shape
- cutting fluid

• We previously considered the log-log scale graph of Taylor's tool life equation, but we may also graph it normally to emphasize the effects.



This graph is representative for most reasonable cutting speeds. The velocities at the high and low ranges do not necessarily exhibit the same relationship.

• There are two basic conditions to trade off,
- Low cost - exemplified by low speeds, low mrr, longer tool life
- High production rates - exemplified by high speeds, short tool life, high mrr
*** There are many factors in addition to these, but these are the most commonly considered



• A simplified treatment of the problem is given below for optimizing cost, First lets look at costs for a cutting tool over the life of a tool,

$$C_t = c_1 + c_2 + c_3$$

where,

C_t = cost per cutting edge
c_1 = the cost to change a tool
c_2 = the cost to grind a tool per edge
c_3 = the cost of the tool per edge

and,

$$c_1 = t_1 \times R_c$$

$$c_2 = t_2 \times \frac{R_s}{N_1}$$

$$c_3 = \frac{C_T}{N_1 \times (N_2 + 1)}$$

where,

t_1 = tool change time
t_2 = tool grind time in minutes
R_c = cutting labour + overhead cost
R_s = grinding labor + overhead cost
C_T = cost of the original tool
N_1 = the number of cutting edges to grind
N_2 = the maximum number of regrinds

and,

$$C_c = R_c \times T$$

where,

C_c = cutting operation cost over life of tool, per edge
T = tool life

Next, lets consider the effects of metal removal rate,

$$Q_T = V \times T \times f \times c \tag{1}$$

where,

Q_T = metal removal rate per edge
V = cutting velocity
f = tool feed rate
c = depth of width of the cut

consider the life of the tool,

$$V \times T^n = C \text{ (Taylors tool life equation)}$$

$$\therefore V = \frac{C}{T^n} \tag{2}$$

Now combine tool life (2) with the mrr (1),

$$Q_T = V \times T \times f \times c = \frac{C}{T^n} \times T \times f \times c = \frac{C \times f \times c}{T^{n-1}}$$

At this point we have determined functions for cost as a function of tool life, as well as the metal removal rates. We can now proceed to find cost per unit of material removed.

$$C_u = \frac{C_c + C_t}{Q_T} = \frac{T^{n-1}}{C \times f \times c}(R_c \times T + C_t)$$

Using some basic calculus, we can find the minimum cost with respect to tool life.

$$\frac{dC_u}{dT} = \left(\frac{1}{C \times f \times c}\right)(R_c \times n \times T^{n-1} + C_t \times (n-1) \times T^{n-2}) = 0$$

$$\therefore R_c \times n \times T = -C_t \times (n-1)$$

$$\boxed{\therefore T = \frac{-C_t \times (n-1)}{R_c \times n} = \frac{C_t}{R_c}\left(\frac{1-n}{n}\right)}$$

• We can also look at optimizing production rates, There are two major factors here when trying to increase the mrr. We can have a supply of tools by the machine, and as the tools require replacement, the only down-time involved is the replacement of the tool.

This gives us an average rate of production,

$$R_p = \frac{Q_T}{T + t_1}$$

where,

R_p = average rate of production

recall from before that,

$$Q_T = \frac{Cfc}{T^{n-1}}$$

now substituting in gives,

$$R_p = \frac{\left(\frac{Cfc}{T^{n-1}}\right)}{T + t_1} = Cfc(T^n + t_1)^{-1}$$

We can now optimize the production rate,

$$\frac{dR_p}{dT} = Cfc[-(T^n + t_1)^{-2} + (nT^{n-1} + t_1)(T^n + t_1)^{-1}] = 0$$

$$\therefore (T^n + t_1)^{-2} = (nT^{n-1} + t_1)(T^n + t_1)^{-1}$$

$$\therefore 1 = (nT^{n-1} + t_1)(T^n + t_1)$$

$$\therefore 1 = nT^{2n-1} + nt_1T^{n-1} + t_1T^n + t_1^2$$

$$\therefore \log(1) = \log(nT^{2n-1}) + \log(nt_1T^{n-1}) + \log(t_1T^n) + \log(t_1^2)$$

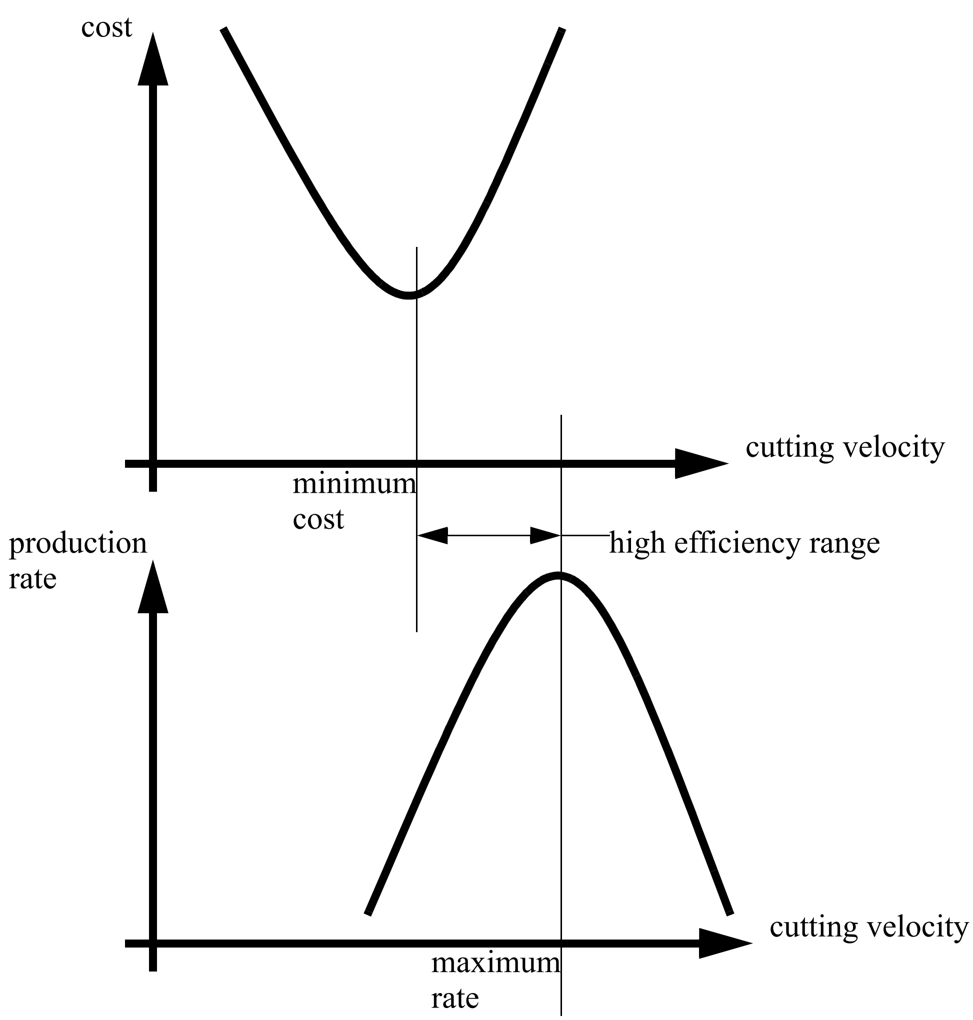
$$\therefore 0 = \log(n) + (2n-1)\log(T) + \log(nt_1) + (n-1)\log(T) + \log(t_1) + n\log(T) + \log(t_1^2)$$

$$\therefore 0 = 2\log(n) + (4n-2)\log(T) + 4\log(t_1)$$

$$\boxed{\therefore \log(T) = \frac{\log(n) + 2\log(t_1)}{1 - 2n}}$$

• We can now put the two optimums in perspective,

Since, $t_1 < C_t/R_c$ then tool life for maximum production is less than economical tool life and as a result, cutting velocity for maximum production is > velocity for lowest cost



7.5 Multivariable Searches

• Local Search Space

• A topographical map shows the relationship between search parameters and cost values.

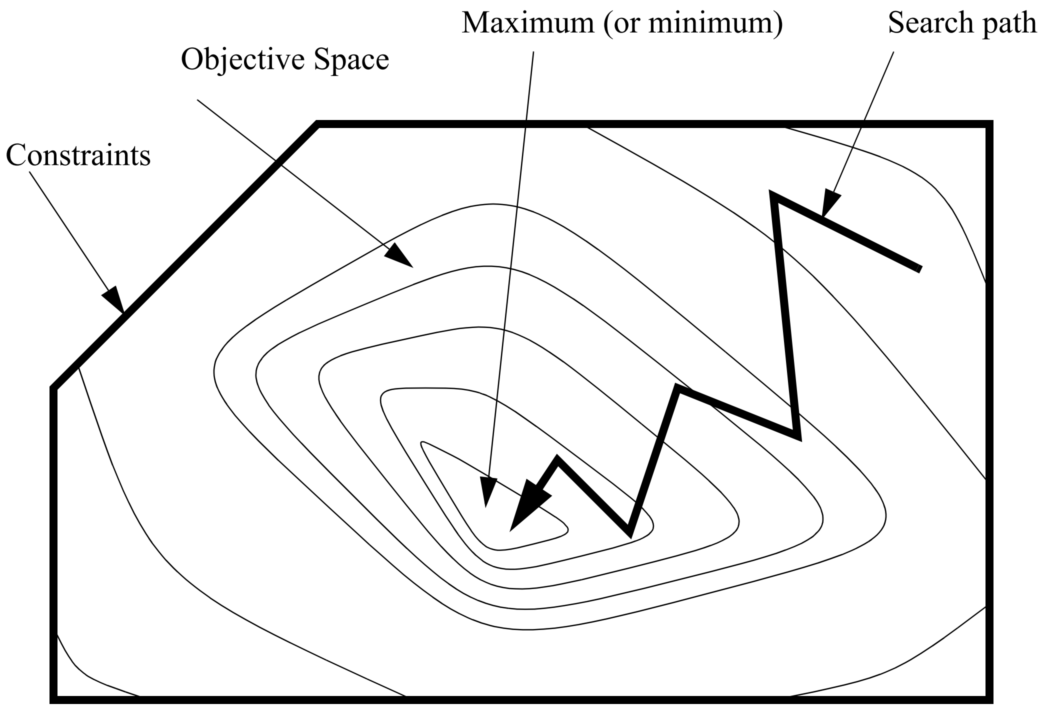


Figure 1.35 Local searches

• Global Search space. In this case the system becomes 'stuck' in a local mimina.

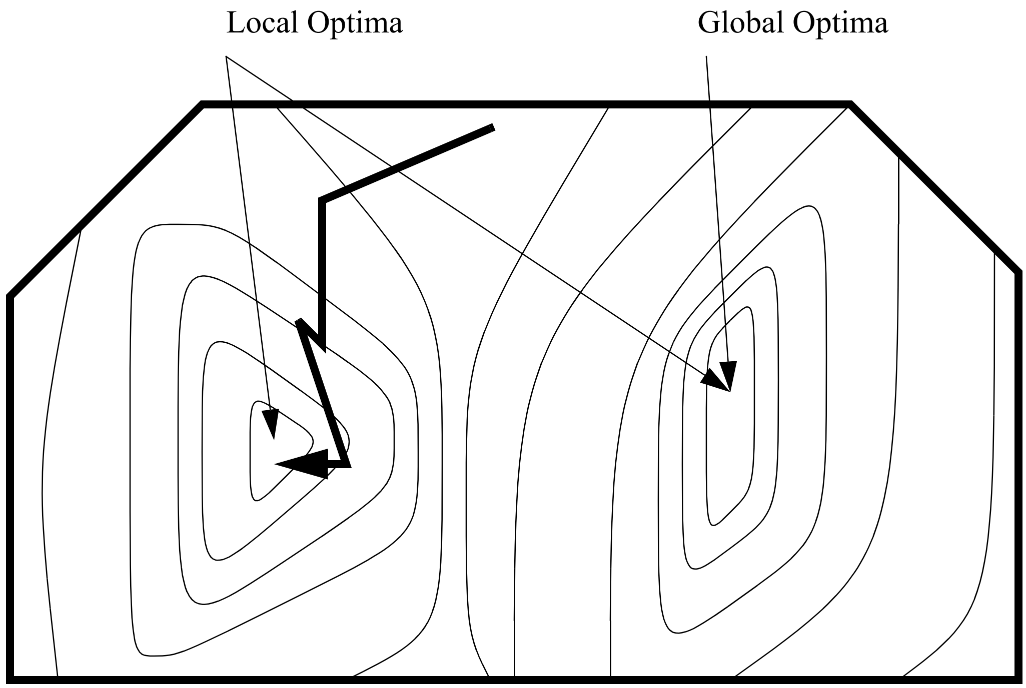


Figure 1.36 Global searches

• Global Search space. In this case the system searches all maxim.

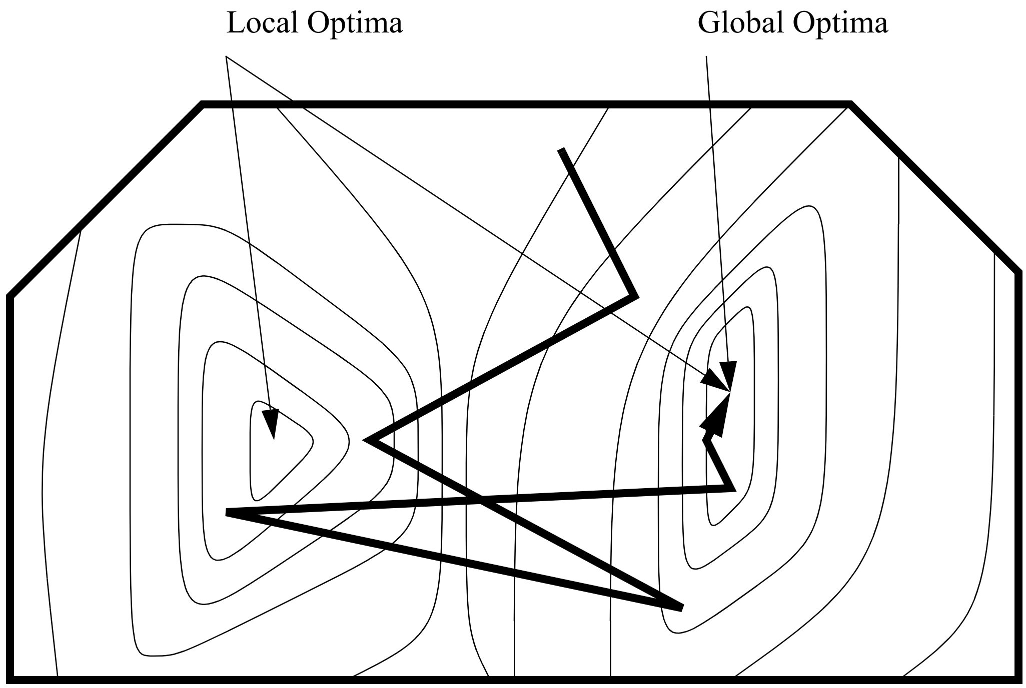


Figure 1.37 A Global Search

7.5.1 Algorithms

• The search algorithms change system parameters and try to lower system parameters.

• The main question is how to change the system paramters to minimize the system value.

7.5.2 Random Walk

7.5.3 Gradient Decent

7.5.4 Simplex

1.6 Summary

•

1.7 Problems

1. A production facility molds plastic parts. Ideally these parts are made with new plastic pellets. However rejected plastic parts are ground into (regrind) pellets and mixed in with new plastic pellets. New plastic costs $1.00 per pound. If a part is discarded (not reground) the total cost of the material is lost. To regrind and dry scrap parts there is a cost of $0.10 per pound. The customer demands that the percentage of regrind cannot exceed 30%. Statistical data was used to find the following relationship between the percentage of regrind and the scrap rate. Write a program to determine the optimum quantity of regrind to minimize the cost per part.

$$S = 2.0 + 3.0 \times 2^{\frac{R+5}{4}}$$

where,

$R =$ percentage of reground material [0, 100%]

$S =$ percentage of parts that are scrap

1.8 Challenge Problems

1. Rocket Fuel Burn

2. Power plant fuel mixture

3. Write a program that recommends the optimum cutting speed for a machining process. Model the program on the example in the chapter.

2. PROJECT SCHEDULING

Topics:

-

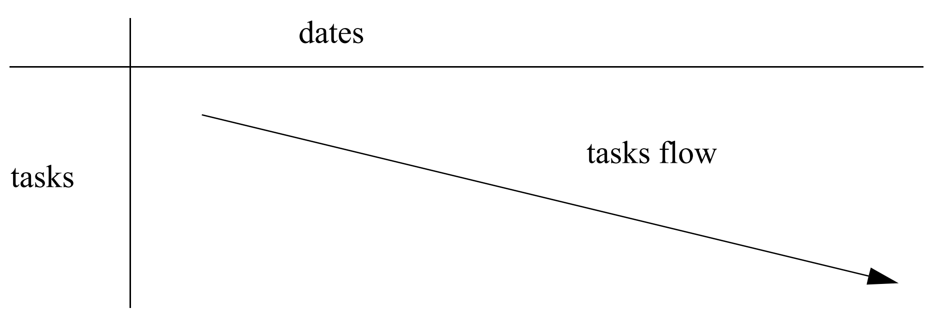
Objectives:

-

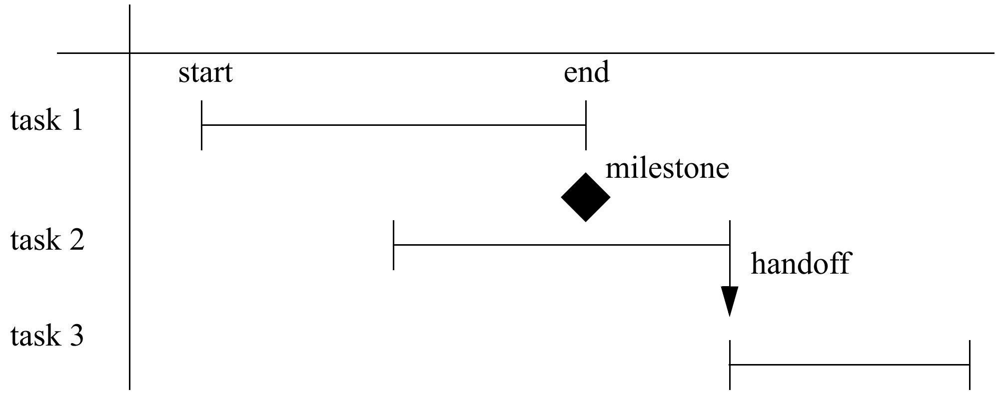
2.1 Introduction

2.2 Gantt Charts

• General form,



• tasks



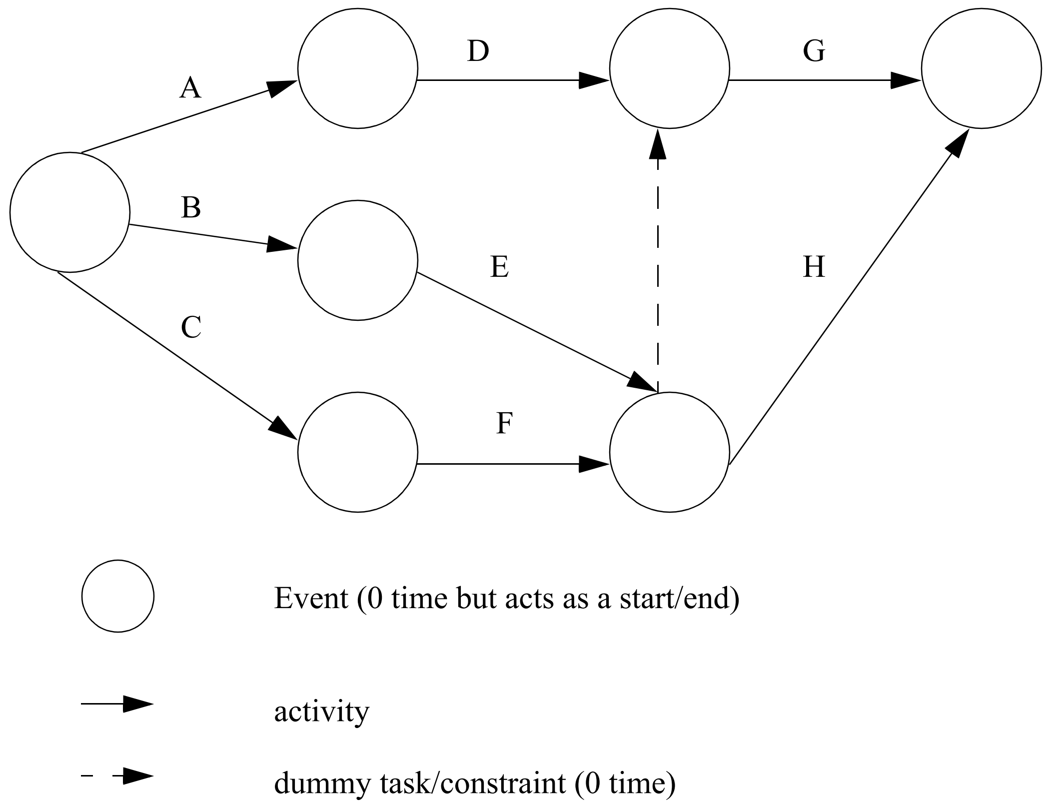
• These charts should be updated on a regular basis to track the progess and completion of each task.

• Things to look for when doing Gannt charts include,
  - gaps when no tasks are boing done
  - too many concurrent tasks
  - too much/too little detail (charts can be broken into subcharts to isolate detail)
  - associate people and resources to tasks

• Good ideas when constructing Gantt Charts,
  - identify critical paths and move forware to create slack time
  - delay costly components to reduce WIP

2.3 Critical Path Method (CPM)

• Tasks (possibly from a Gannt chart) can be put in a network diagram.



• Earliest Start (ES) the earliest time a task can start.

• Minimum Project Duration - the shortest time to complete the project based upon the longest path.

• Desired Project Duration - the planned time for completing the project.

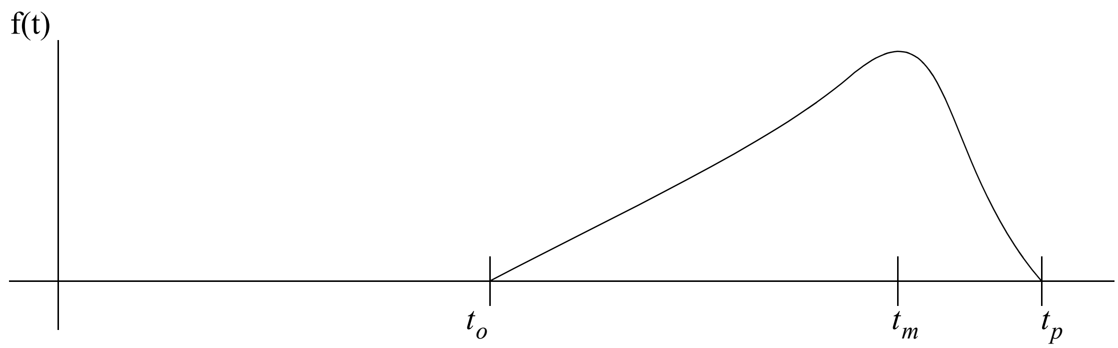
• Latest Start (LS) - the latest a task can start for a Desired Project Duration.

• Total Float (TF) = LS - ES

• Critical Path - the sequence of tasks that take the longest, and dictate the Minimum Project Duration.

2.4 Program Evaluation and Review Technique (PERT)

• In CPM we assume that each activity has a fixed time, in practice the task lengths vary. This variation can be shown with the Beta distribution.



where,

t_o = optimistic time estimate

t_m = most likely time estimate

t_p = pessimistic time estimate

• task times are expressed with three numbers separated by dashes to-tm-tp to represent task times.

• the mean (effective) time can be found with,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

• A Standard Deviation for each activity time can be approximated using,

$$\sigma_i = \left| \frac{t_{p_i} - t_{o_i}}{6} \right|$$

• The te valuaes can be used to do a CPM analysis of a network diagram. Once the Critical Path is identified the overal task time and variance can be calculated using,

$$\sigma_T = \sqrt{\sum \sigma_i^2}$$

$$T_e = \sum t_{e_i}$$

• To find the chance that the process will be done by the time Ts, the z value can be calculated. The z value can then be used to find the probability of completion using the cummulative normal distribution function.

$$z = \frac{T_s - T_e}{\sigma_T}$$

2.5 Problems

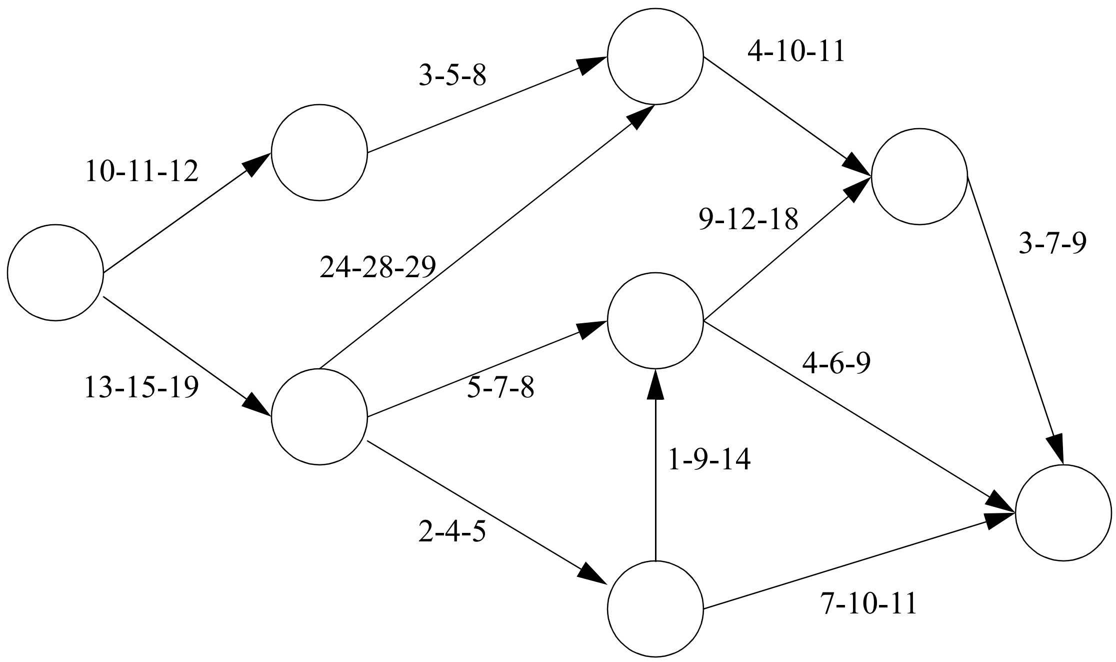
1. A new building is being constructed and the following tasks are required. The normal workdays are 7am-3pm, Monday to Friday. Overtime is possible, however the costs make it highly underiable. Write a Gantt chart for completion of the job in 3 months.
   Site Preparation - 1 month
   Foundations - 1 month - after Site Preparation
   Framing - 2 weeks - after Foundations
   Plumbing - 5 weeks - after Framing
   Electrical - 6 weeks - after Framing
   Inspection - after Plumbing and Electrical
   Drywall - 2 weeks - after Inspection
   Painting - 1 week - after Drywall
   Hardware - 1 week - after Painting
   Carpet - 3 days - after Painting

2. Develop a project activity network for problem 1.

3. Identiy the critical path for problem 2.

4. Consider the PERT network diagram below and find the likelyhood that the project will be

complete in 40 days.



5. For problem 4, find a target completion date for the project that will make it 50% likely that it will be complete.

2.6 Challenge Problems

3. DIRECTED GRAPHS

Topics:

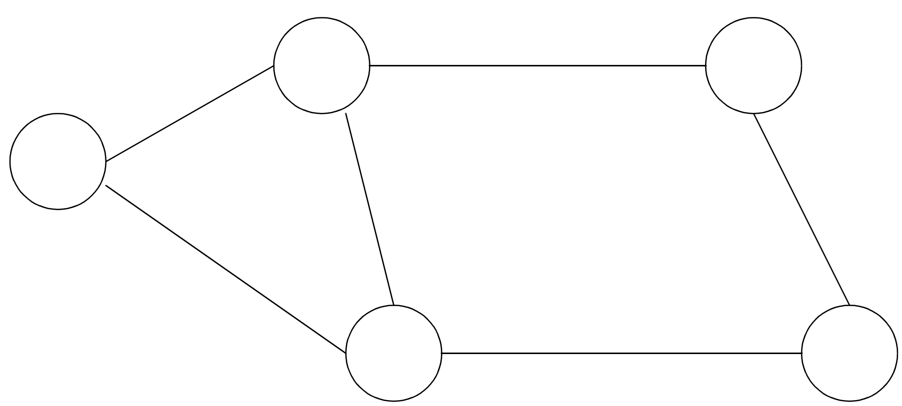
-

Objectives:

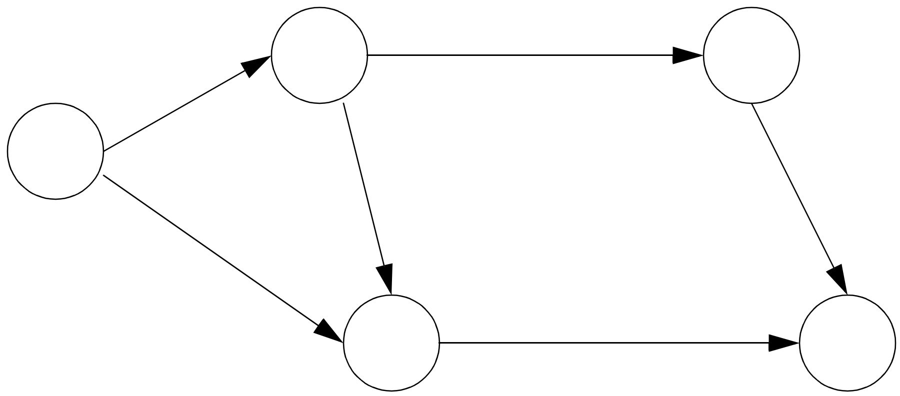
-

3.1 Introduction

• Graphs are normally data structures that have vertices (circles) and edges (lines) as shown below.



• In a directed graph the edges will have a direction assigned, as shown below.



3.2 Data Structures

• A simple data structure for representing a graph is shown below.

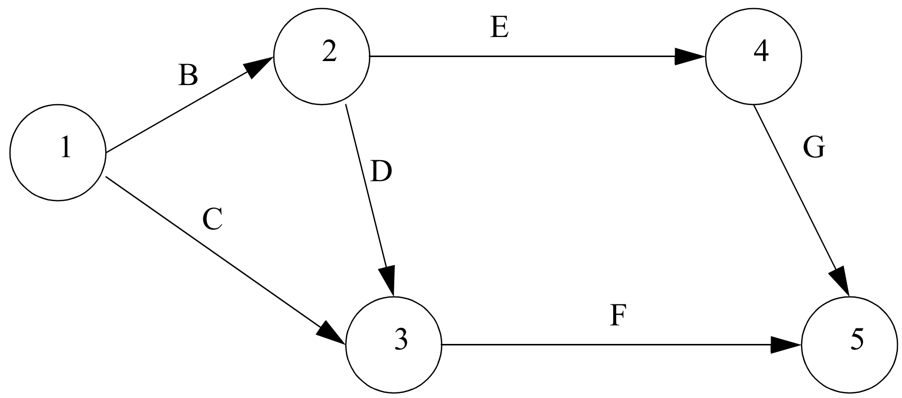
Vertex list:

Vertex

Edge list:

Edge	From	To

• Consider the graph below.



Vertex list:

Vertex
1
2
3
4
5

Edge list:

Edge	From vert.	To vert.
B	1	2
C	1	3
D	2	3
E	2	4
F	3	5
G	4	5

• Representing a graph in Scilab

```
function foo = find_index(_list, name) // a function to find list numbers given names
	foo = -1; // use -1 to indicate no match found yet
	A = size(_list); // find the rows and columns in the list
	cnt = A(1, 1); // get the rows in the list
	for i = 1 : cnt, // loop through the rows
				if name == _list(i) then // look for a name match
						foo = i; // record the matching row number
						break; // no point continueing the for loop
				end,
	end
	if foo == -1 then mprintf("ERROR: list name %s not found\n", name); end
endfunction

vertex_names = ["1" ; "2" ; "3" ; "4" ; "5"]; // define the vertices

function foo = vert_number(name) // a function to find vertex numbers given names
	foo = find_index(vertex_names, name);
endfunction

edge_names = ["B" ; "C" ; "D" ; "E" ; "F" ; "G"]; // define the edges.

function foo = edge_number(name) // a function to find edge numbers given names
	foo = find_index(edge_names, name);
endfunction

edge = [edge_number("B"), vert_number("1"), vert_number("2")];
edge = [edge ; [edge_number("C"), vert_number("1"), vert_number("3")]];
edge = [edge ; [edge_number("D"), vert_number("2"), vert_number("3")]];
edge = [edge ; [edge_number("E"), vert_number("2"), vert_number("4")]];
edge = [edge ; [edge_number("F"), vert_number("3"), vert_number("5")]];
edge = [edge ; [edge_number("G"), vert_number("4"), vert_number("5")]];
```

• Representing a graph in C

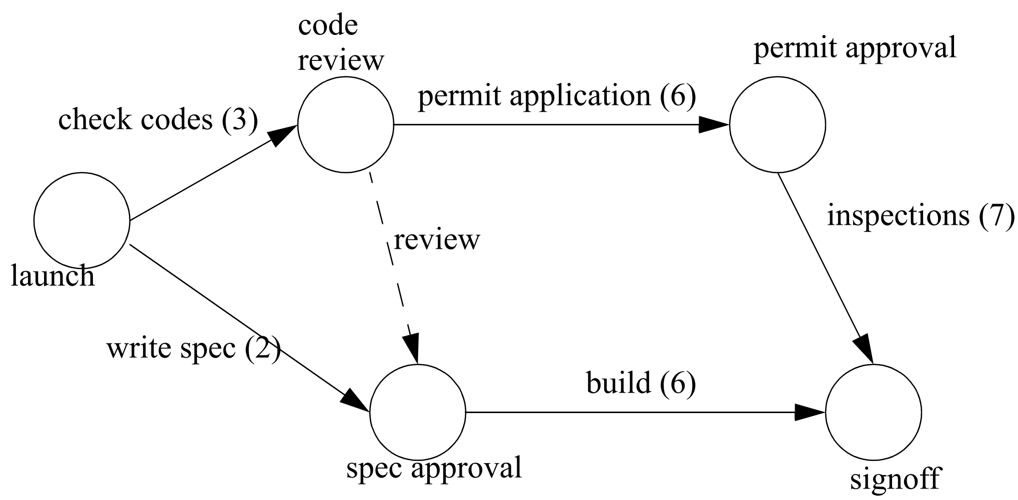
```
int  find_number(char list[][], char *name){
       int                     i;
       if(name == NULL) return -1;
       for(i = 0; list[i][0] != NULL; i++){
                               if(strcmp(name, list[i]) == 0){
                                               return i;
                               }
       }
       printf("ERROR: search name %s not found \n", name);
       return -1;
}

char vert_names[][] = {{"1"}, {"2"}, {"3"}, {"4"}, {"5"}, NULL};
int  vert_number(char *name){ return find_number(vert_names, name);}

char edge_names[][] = {{"B"}, {"C"}, {"D"}, {"E"}, {"F"}, {"G"}, NULL};
int  edge_number(char *name){ return find_number(edge_names, name);}

#define MAX_EDGES              10
int     edge_cnt = 0;
int     edges[MAX_EDGES][3];

void    add_edge(char *edge, char *from_vert, char *to_vert){
        edges[edge_cnt][0] = edge_number(edge);
        edges[edge_cnt][1] = vert_number(from_vert);
        edges[edge_cnt][2] = vert_number(to_vert);
        edge_cnt++;
}

int main(){
        add_edge("B", "1", "2");
        add_edge("C", "1", "3");
        add_edge("D", "2", "3");
        add_edge("E", "2", "4");
        add_edge("F", "3", "5");
        add_edge("G", "4", "5");

        return 0;
}
```

3.3 Applications

3.3.1 Precedence Identification

• To determine a precedence we add a precedence count for each vertex. The basic check is,
1. Set all the precendence values to 1 (lowest precedence)
2. Cycle through and check each edge. For each edge ensure that the ’to’ vertex has a precedence that is at least one greater than the ’from vertex.

• Consider a Network Diagram for a CPM problem.



• The algorithm implemented in Scilab (assuming the 'find_index' function is the same as before)

```
vert_names = ["launch" ; "code review" ; "spec approval" ; "permit approval" ; "signoff"]; // vertices
A = size(vert_names); // find the rows and columns in the list
vert_cnt = A(1, 1); // get the rows in the list

edge_names = ["check codes" ; "write spec" ; "permit application" ; "review" ; "build" ; "inspections"]; /
/ define the edges.
A = size(edge_names); // find the rows and columns in the list
edge_cnt = A(1, 1); // get the rows in the list

//------ Previous Code ----------------------

function foo = find_index(_list, name) // a function to find list numbers given names
	foo = -1; // use -1 to indicate no match found yet
	A = size(_list); // find the rows and columns in the list
	cnt = A(1, 1); // get the rows in the list
	for i = 1 : cnt, // loop through the rows
				if name == _list(i) then // look for a name match
						foo = i; // record the matching row number
						break; // no point continueing the for loop
				end,
	end
	if foo == -1 then mprintf("ERROR: list name %s not found\n", name); end
endfunction

function foo = vert_number(name) // a function to find vertex numbers given names
	foo = find_index(vert_names, name);
endfunction
function foo = edge_number(name) // a function to find edge numbers given names
	foo = find_index(edge_names, name);
endfunction

//-------------------------------------------------

// Note: an extra column is added to include the time for each activity
edge = [edge_number("check codes"), vert_number("launch"), vert_number("code review"), 3];
edge = [edge ; [edge_number("write spec"), vert_number("launch"), vert_number("spec approval"), 2]];
edge = [edge ; [edge_number("permit application"), vert_number("code review"), vert_number("permit
approval"), 6]];
edge = [edge ; [edge_number("review"), vert_number("code review"), vert_number("spec approval"), 0]];
edge = [edge ; [edge_number("build"), vert_number("spec approval"), vert_number("signoff"), 6]];
edge = [edge ; [edge_number("inspections"), vert_number("permit approval"), vert_number("signoff"), 7]];

// initialize all precendence values to 1
precedence = [1];
for i = 2 : vert_cnt,
	precedence = [precedence ; 1];
end

// Loop through and update precendences
for i = 1 : edge_cnt - 1,
	for j = 1 : edge_cnt,
				prev_vert = edge(j, 2); // find the previous and current vertices
				next_vert = edge(j, 3);
				prev_precedence = precedence(prev_vert); // find the precendence of
previous/next vertices
				next_precedence = precedence(next_vert);
				if (prev_precedence + 1) > next_precedence then // check to see if the
precedence needs to be updated
						precedence(next_vert) = prev_precedence + 1;
				end,
	end,
end
```

3.3.2 Graph Searching

• Identifying the vertex costs.

```
// Assumes the code for vertices has already been defined and run.

// initialize all vertex cost values to 0
cost = [0];
for i = 2 : vert_cnt,
        cost = [cost ; 0];
end

maximum_cost = 0;
last_vert = 0;

// Loop through and update precendences
for i = 1 : edge_cnt - 1,
        for j = 1 : edge_cnt,
                                prev_vert = edge(j, 2); // find the previous and current vertices
                                next_vert = edge(j, 3);
                                edge_cost = edge(j, 4);
                                prev_cost = cost(prev_vert); // find the cost of previous/next vertices
                                next_cost = cost(next_vert);
                                if (prev_cost + edge_cost) > next_cost then // check to see if the cost
needs to be updated
                                                cost(next_vert) = prev_cost + edge_cost;
                                end,
                                if cost(next_vert) > maximum_cost then
                                                maximum_cost = cost(next_vert);
                                                last_vert = next_vert;
                                end
        end,
end

mprintf("The critical path cost is %f \n", maximum_cost);
```

3.4 Other Topics

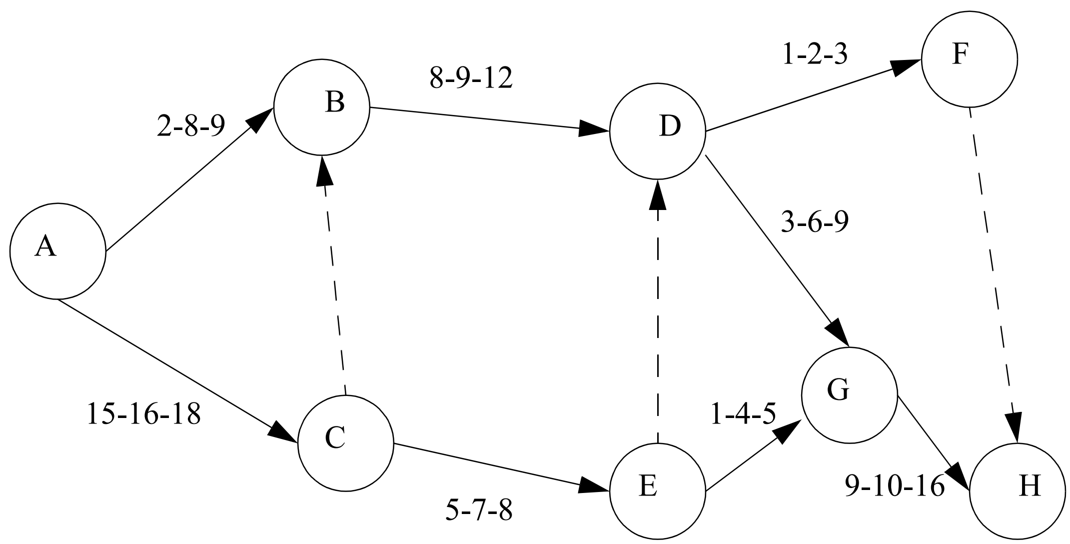
• ADD: resource limited plans

1.5 Summary

•

1.6 Problems

1. Write a general program that will accept a network diagram entered as vertices and arcs. The program will then do a PERT analysis to determine the Te value and the standard deviation. Test the program with the following network diagram.



(ans. Te=42.333, S.D.=1.826)

1.7 Challenge Problems

2. TREE DATA STRUCTURES

Topics:

•

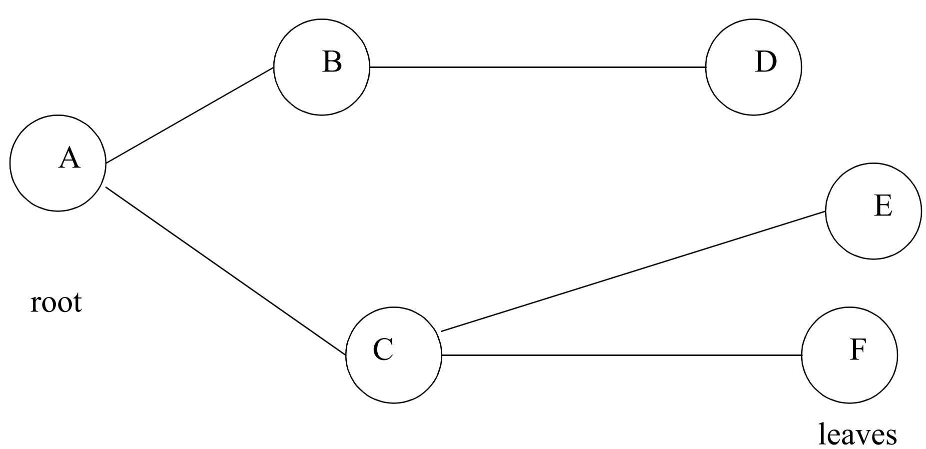
Objectives:

•

2.1 Introduction

• Trees are a special case of a graph data structure. The connections radiate out from a single root without cross connections.

• The tree has nodes (shown with circles) that are connected with branches. Each node will have a parent node (except for the root) and may have multiple child nodes.



- In an unordered tree the children of a node can be unordered, however in an ordered tree the children take priority, often by listing them in order.

- A Boolean tree applies operators for each of the nodes

2.2 Data Structures

- A simple data structure for representing a tree is shown below.

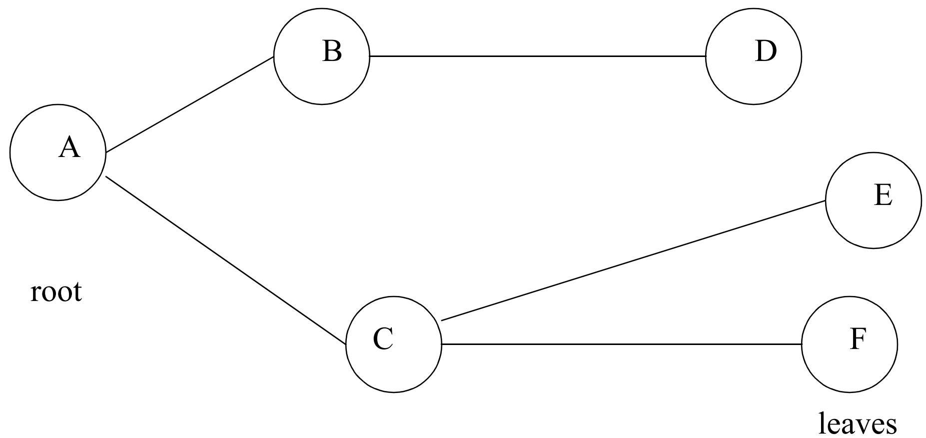
Node list:

Node

Branch list:

Parent	Child

• Consider the graph below.



Node #	Node Name
1	A
2	B
3	C
4	D
5	E
6	F

Parent	Child
1	2
1	3
2	4
3	5
3	6

• Representing a tree in Scilab

```
function foo = find_index(_list, name) // a function to find list numbers given names
       foo = -1; // use -1 to indicate no match found yet
       A = size(_list); // find the rows and columns in the list
       cnt = A(1, 1); // get the rows in the list
       for i = 1 : cnt, // loop through the rows
                              if name == _list(i) then // look for a name match
                                               foo = i; // record the matching row number
                                               break; // no point continueing the for loop
                              end,
       end
       if foo == -1 then mprintf("ERROR: list name %s not found\n", name); end
endfunction

node_names = ["A" ; "B" ; "C" ; "D" ; "E" ; "F"]; // define the nodes

function foo = node_number(name) // a function to find node numbers given names
       foo = find_index(node_names, name);
endfunction

branch = [node_number("A"), node_number("B")];
branch = [branch ; [node_number("A"), node_number("C")]];
branch = [branch ; [node_number("B"), node_number("D")]];
branch = [branch ; [node_number("C"), node_number("E")]];
branch = [branch ; [node_number("C"), node_number("F")]];
```

• Representing a graph in C

```
int  find_number(char list[][], char *name){
       int                    i;
       if(name == NULL) return -1;
       for(i = 0; list[i][0] != NULL; i++){
                              if(strcmp(name, list[i]) == 0){
                                             return i;
                              }
       }
       printf("ERROR: search name %s not found \n", name);
       return -1;
}

char node_names[][] = {{"A"}, {"B"}, {"C"}, {"D"}, {"E"}, {"F"}, NULL};
int  node_number(char *name){ return find_number(node_names, name);}


#define MAX_BRANCHES           10
int    branch_cnt = 0;
int    branch[MAX_BRANCHES][2];

void   add_branch(int parent, int child){
       branch[branch_cnt][0] = node_number(parent);
       branch[branch_cnt][1] = node_number(child);
       branch_cnt++;
}

int main(){
       add_branch("A", "B");
       add_branch("A", "C");
       add_branch("B", "D");
       add_branch("C", "E");
       add_branch("C", "F");

       return 0;
}
```

2.3 Applications

2.3.1 Precedence Identification

• To determine a precedence we add a precedence count for each node. The basic check is,

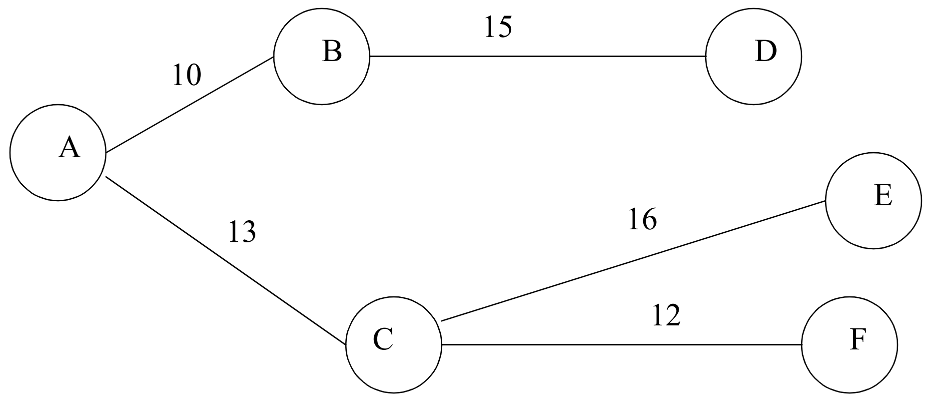
1. Set all the precedence values to 1 (lowest precedence)
2. Cycle through and check each branch. For each branch ensure that the child vertex has a precedence that is one greater than the parent vertex.

• Consider the previous example.

```
function foo = find_index(_list, name) // a function to find list numbers given names
       foo = -1; // use -1 to indicate no match found yet
       A = size(_list); // find the rows and columns in the list
       cnt = A(1, 1); // get the rows in the list
       for i = 1 : cnt, // loop through the rows
                               if name == _list(i) then // look for a name match
                                                foo = i; // record the matching row number
                                                break; // no point continueing the for loop
                               end,
       end
       if foo == -1 then mprintf("ERROR: list name %s not found\n", name); end
endfunction

node_names = ["A" ; "B" ; "C" ; "D" ; "E" ; "F"]; // define the nodes
node_cnt = 6; // get the rows in the list

function foo = node_number(name) // a function to find node numbers given names
       foo = find_index(node_names, name);
endfunction

branch = [node_number("A"), node_number("B")];
branch = [branch ; [node_number("A"), node_number("C")]];
branch = [branch ; [node_number("B"), node_number("D")]];
branch = [branch ; [node_number("C"), node_number("E")]];
branch = [branch ; [node_number("C"), node_number("F")]];
branch_cnt = 5;

for i=1:node_cnt, // set all precendence values to 1
       precedence(i) = 1;
end

for i=1:node_cnt-1, // loop through and update precendence values
       for j=1:branch_cnt,
                               if precedence(branch(j,1)) >= precedence(branch(j,2)) then
                                                precedence(branch(j,2)) = precedence(branch(j,1)) + 1;
                               end
       end
end
```

2.3.2 Tree Searching

• Assume that each branch has a cost associated. We can then find the cost of each of the nodes.



• Identifying the vertex costs.

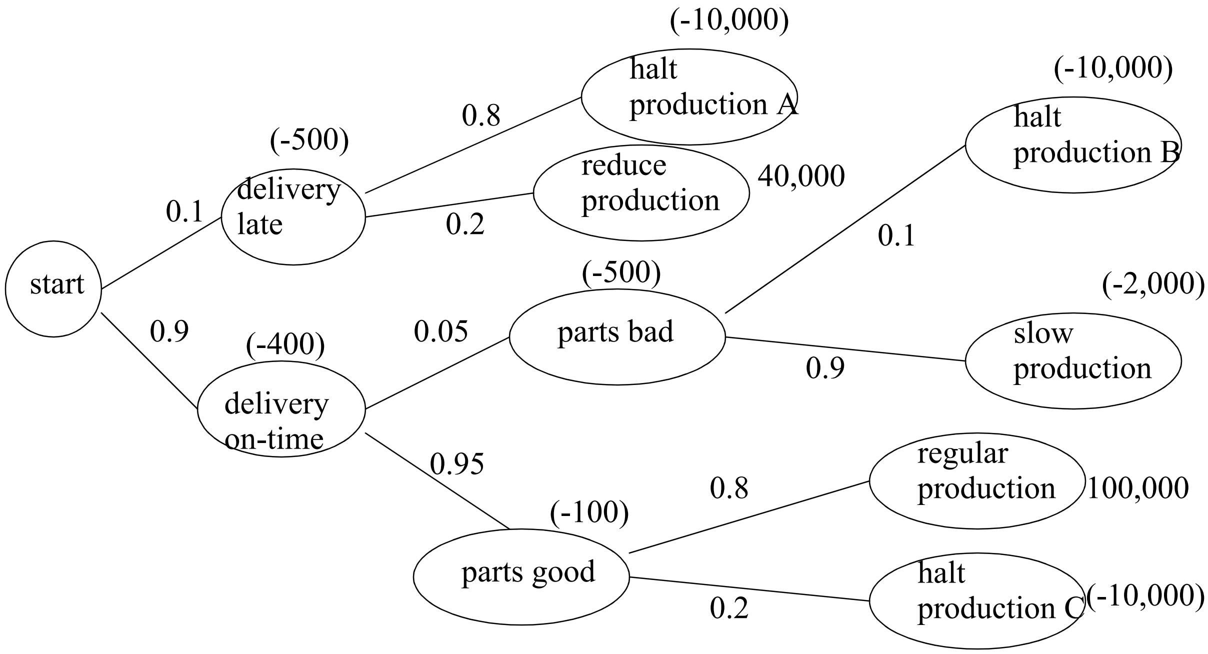
```
function foo = find_index(_list, name) // a function to find list numbers given names
       foo = -1; // use -1 to indicate no match found yet
       A = size(_list); // find the rows and columns in the list
       cnt = A(1, 1); // get the rows in the list
       for i = 1 : cnt, // loop through the rows
                               if name == _list(i) then // look for a name match
                                                foo = i; // record the matching row number
                                                break; // no point continueing the for loop
                               end,
       end
       if foo == -1 then mprintf("ERROR: list name %s not found\n", name); end
endfunction

node_names = ["A" ; "B" ; "C" ; "D" ; "E" ; "F"]; // define the nodes
node_cnt = 6; // get the rows in the list

function foo = node_number(name) // a function to find node numbers given names
       foo = find_index(node_names, name);
endfunction

branch = [node_number("A"), node_number("B"), 10];
branch = [branch ; [node_number("A"), node_number("C")], 13];
branch = [branch ; [node_number("B"), node_number("D")], 15];
branch = [branch ; [node_number("C"), node_number("E")], 16];
branch = [branch ; [node_number("C"), node_number("F")], 12];
branch_cnt = 5;

for i=1:node_cnt, // set all node cost values to 0
       cost(i) = 0;
end

for i=1:node_cnt-1, // loop through and update cost values
       for j=1:branch_cnt,
                               cost(branch(j, 2)) = cost(branch(j, 1)) + branch(j, 3);
       end
end
```

1.4 Summary

•

1.5 Problems

1. 1. Consider the following probability tree for a single day in a production facility. There is a 10% chance that 'delivery late' will occur, and an 90% chance that 'delivery on-tine' will occur. The values on the branches indicate the probability of an event occurring. For example, if there is 'delivery on-time' there will be a net cost of $400 to process the incoming parts. Write a program that will use the probabilities and costs to calculate the effective income (expenses) for the facility.



(ans. 66046)

1.6 Challenge Problems