Using Geometry to Estimate π

Let's start with a simple approximation, using an inscribed hexagon to approximate the area of the prescribed circle. A hexagon is used because of its special attributes, i.e., the length of the sides of an inscribed hexagon equal the radius of the prescribed circle.

So, using a **unit circle** (R=1), in the simple diagram shown below, we create the hexagon $ABCDEF$ where the side of the hexagon S=R, and we have:

$$Side\ of\ hexagon = S = \overrightarrow{EF} = Radius\ of\ circle = R = \overrightarrow{OK} = 1$$

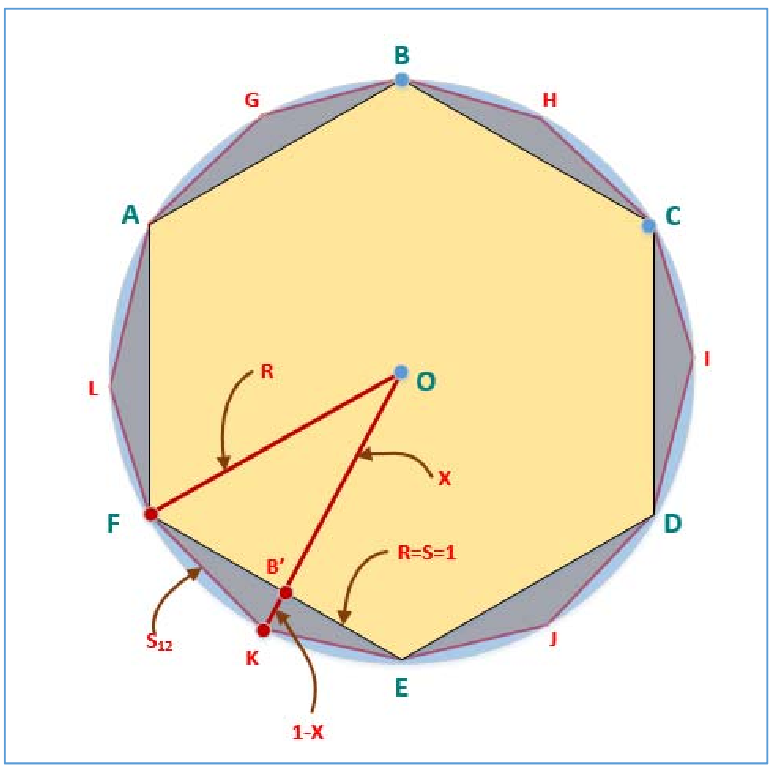
$$Perim_{cir} \approx Perim_6 = 6S = 6R$$

$$Diam_{cir} = Diam_6 = 2R$$

$$\pi \approx \frac{Perim_{cir}}{Diam_{cir}} \approx \frac{Perim_6}{Diam_6} \approx \frac{6R}{2R} \approx 3$$

Where:

$Perim_{cir} = Perimeter\ of\ circle$
$Perim_6 = Perimeter\ of\ hexagon$
$Diam_{cir} = Diameter\ of\ circle$
$Diam_6 = Diameter\ of\ hexagon$



So, the first approximation of π using this method is 3.

Now, let's create the vertices of an inscribed dodecahedron by using bisecting radii, like $\overrightarrow{OK}$, which divides the side of the hexagon $\overrightarrow{EF}$ into two equal segments, $\overrightarrow{FB'}$ and $\overrightarrow{EB'}$.

Considering the right triangle $OB'F$, and using the Pythagorean theorem, we have:

$$R^2 = x^2 + (B'F)^2$$

$$x^2 = R^2 - (B'F)^2$$

$$x^2 = 1^2 - \left(\frac{EF}{2}\right)^2 = 1 - \left(\frac{S}{2}\right)^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$x = \sqrt{\frac{3}{4}}$$

And, considering the triangle $B'KF$, we have:

$${S_{12}}^2 = (1 - x)^2 + (B'F)^2$$

$${S_{12}}^2 = \left(1 - \sqrt{\frac{3}{4}}\right)^2 + (B'F)^2 = \left(1 - \sqrt{\frac{3}{4}}\right)^2 + \left(\frac{1}{2}\right)^2 = \left(1 - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}$$

$$S_{12} = \sqrt{\left(1 - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}}$$

Where: Subscript 12 indicates dodecahedron

And the new (better) approximation of π now is:

$$\pi \approx \frac{Perim_{cir}}{Diam_{cir}} \approx \frac{Perim_{12}}{Diam_{12}} = \frac{12\left[\sqrt{\left(1 - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}}\right]}{2R} = \frac{12\left[\sqrt{\left(1 - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}}\right]}{2} = 6\left[\sqrt{\left(1 - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}}\right]$$

π =3.105828541230248

Using the above equations and doubling the number of vertices of the inscribed polygon each time, Table-1 summarizes the approximated values of π using up to a polygon of more than 100,000,000 vertices (see below). The values in the table were generated using a simple routine in scilab. The scilab code is provided following the table. This method is good for demonstrating the ability to estimate π from first principles, but cannot come anywhere near the numeric solution approximations using series, employed by scientists nowadays, where they can calculated π to more than a trillion decimal places.

Table-1, Geometric Approximation of π

Number of Vertices of Inscribed Polygon	Length of Polygon Side	Estimate of π
6	1	3
12	0.51763809020504	3.105828541230248
24	0.26105238444010	3.132628613281237
48	0.13080625846029	3.139350203046866
96	0.06543816564355	3.141031950890509
192	0.03272346325297	3.141452472285462
384	0.01636227920787	3.141557607911857
768	0.00818120805247	3.141583892148318
1,536	0.00409061258233	3.141590463228050
3,072	0.00204530736068	3.141592105999271
6,144	0.00102265381403	3.141592516692157
12,288	0.00051132692372	3.141592619365384
24,576	0.00025566346395	3.141592645033691
49,152	0.00012783173224	3.141592651450768
98,304	0.00006391586615	3.141592653055037
196,608	0.00003195793308	3.141592653456104
393,216	0.00001597896654	3.141592653556371
786,432	0.00000798948327	3.141592653581438
1,572,864	0.00000399474164	3.141592653587704
3,145,728	0.00000199737082	3.141592653589271
6,291,456	0.00000099868541	3.141592653589663
12,582,912	0.00000049934270	3.141592653589761
25,165,824	0.00000024967135	3.141592653589785
50,331,648	0.00000012483568	3.141592653589791
100,663,296	0.00000006241784	3.141592653589793

Scilab Code used to generate Table-1

```
// ---------------------------------------
// Estimate pi using inscribed polygons
//----------------------------------------

clc
// Set basic starting parameters
n=6  // Starting number of vertices
Actina=1  // Radius, always 1, unit circle
Plevra=1  // Staring length of side.  For inscribed hexagon, Side=Radius
Perimetros=n*Plevra  // Perimeter
Diametros=2*Actina  // Diameter
myPI=Perimetros/Diametros  // First estimate of pi

// print header and first row of pi estimate
mprintf('     Num     Actina      Plevra     Perimet   Diamet        myPI\n')
mprintf('%10d  %7.4f  %16.14f  %10.8f  %7.4f
%24.20f\n',n,Actina,Plevra,Perimetros,Diametros,myPI)

// Looop a ndiv times, each time doubling the number of vertices.
for ndiv=1:1:24
  n=n*2  // Double # of vertices
  tPlevra=Plevra/2  // half of previous side, Radius bisects
  x=sqrt(Actina^2-tPlevra^2)  // X portion or radius, as indicated on
accompanying drawing
  Plevra=sqrt((1-x)^2+tPlevra^2)  // Length of new side through Pythagorean
  Perimetros=n*Plevra // Perimeter of new inscribed polygon
  myPI=Perimetros/Diametros  // Diameter is diameter of circle, does not
change
  // Print new row of pi estimate and other params
  mprintf('%10d  %7.4f  %16.14f  %10.8f  %7.4f
%24.20f\n',n,Actina,Plevra,Perimetros,Diametros,myPI)
end
```